



SCIENTIFIC BULLETIN
PHYSICAL AND
MATHEMATICAL RESEARCH

ILMIY XABARNOMA
FIZIKA-MATEMATIKA
TADQIQOTLARI

2025
VOLUME 7
ISSUE 2

ЗАХИРИДДИН
МУҲАММАД БОБУР
НОМИДАГИ АНДИЖОН
ДАВЛАТ УНИВЕРСИТЕТИ

ANDIJAN STATE
UNIVERSITY NAMED
AFTER ZAKHIRIDDIN
MUKHAMMAD BABUR

ИЛМИЙ ХАБАРНОМА

Физика-математика
тадқиқотлари

SCIENTIFIC BULLETIN

Physical and
Mathematical Research

2025/№2(7)

Андижон
2025 йил

Муассис

Захириддин Мухаммад Бобур номидаги Андижон давлат университети

**ИЛМИЙ ХАБАРНОМА.
ФИЗИКА-МАТЕМАТИКА
ТАДҚИҚОТЛАРИ**

Журнал бир йилда 2 марта чоп этилади.

Андижон вилояти ахборот ва оммавий коммуникациялар бошқармаси
томонидан 2019-йил 26 декабрда
0452 рақам билан рўйхатга олинган.

Нашр индекси: 344

Нашр учун масъул:
Х.А. Махмудов

Босишга рухсат этилди:
27.12.2019.

Қоғоз бичими: 60x81 1/8

Босма табағи: 13,5

Офсет босма. Офсет қоғози.

Адади: 110 дона.

Баҳоси келишилган нархда.

Буюртъа №: 165.

“Мухаррир” нашриёти матбаа бўлимида чоп этилди.
Тошкент шаҳри, Сўғалли ота кўчаси 7-уй

Таҳририят манзили:

170100, Андижон шаҳри, Университет кўчаси, 129. Телефон: +998911602043.

Факс: (374) 223-88-30

E-mail: adu_xabarnoma@mail.ru Расмий сайт: <https://asujournal.uz/>

Физика	
С.З.Зайнабидинов, Н.А.Тургунов, Ш.К.Акбаров, Р.М.Турманова Н.Б.Хайтимметов Влияние примесных скоплений на оптические свойства кремния, легированного никелем и медью	7
R.Sh.Daminov, J.K.Avloni, A. Turayev, M.S. Qambarova Cost-effective synthesis of graphene dispersions by liquid-phase exfoliation and their structural evaluation	12
A. Turaev, R. Burkhanov Study of kinetics and formation of oxygen precipitates in silicon single crystals	17
B.Q Sultonov Amorf yarimo'tkazgichlar uchun Kubo – Grinvud formulasi	23
O.Kh. Kuldashov, D.O. Nabiev Mathematical model of analysis of causes of cable line damages to improve reliability indicators during operation	27
A.T. Mamadalimov, Sh.B.Utamuradova, N.K.Khakimova, S.U.Umarova, Sh.M.Norbekov, D.E.Xazratov, D.A.Azimova Paxta tolalarining optik xususiyatlarini tadqiq qilish	31
Математика	
J.R.Alieva Mathematical models of immunity diffusion in a semi-conductor and their role in electronic technology.	39
N. Umaraliyeva Discrete game with summary constraints on controls and its animation model	45
D.U.Kushakova, T.T.Ibaydullaev Guaranteed evasion time in a differential game for an infinite system in Hilbert space l_2	51
M.K.Maxkamov, J.D.Dexkonov, A.X.Akbarov Vektorlar yordamida geometrik masalalar yechish.	58
Х.К.Иброхимов О разрешимости краевых задач для вязко-трансзвукового уравнения с несимметричными условиями по времени	61
K.T. Karimov, M.M.Toshmatov Buziladigan oddiy differensial tenglama uchun nolokal shartli spektral masalalar	71
Ю.П.Апаков, Р.А.Умаров Решение краевой задачи для уравнения третьего порядка с переменными коэффициентами	74
Atabaev Odiljon Xusniddin o'g'li O'zgaruvchan zichlikli nochiziqli manba va yutilishga ega ikki karra buziluvchan nochiziqli parabolik masalani sonli tadqiq qilish	84
G.A.Vasiyeva Gyrobifastigium qirralari bo'ylab ikki quvlovchi va bir qochuvchining tutish differensial o'yni	90

A.A.Adahamov, K.S.Kozimjonov Kesik kubda ikkita aktiv bitta sust quvlovchilar va bir qochuvchining differensial o'yini	95
G. A. G'anijonova Kvadrat gumbaz qirralari bo'ylab ikki quvlovchi va bir qochuvchining tutish differensial o'yini	102
R.B. Ortiqova “Burilgan ikki tepalikli shakl” qirralarida uch quvlovchi va bir qochuvchining differensial o'yini	109
Sh. Sh. Maxammadzokirov n burchakli muntazam prizmada 2 quvlovchi va bir qochuvchining tutish differensial o'yini	115
O.S.Akhmedov, D.B.Muzaffarova Estimation of accuracy of the numerical solution of Cauchy problem for quadratic dynamical systems using Runge-Kutta methods	120
I. G'.Yuldashev To'ldiruvchi fazoning o'lchami maksimal nilradikali filiform bo'lgan yechiluvchan Li algebrasining differensiallashlari	127
R.A.Abduraxmonova Differential game of three pursuers and one evader on a rhombicuboctahedron	133
I.M.Yusupov Uch quvuvchi va bir qochuvchili qochish differensial o'yini.	141
К.Т. Каримов, А.М. Шокиров О единственности решения трехмерной задачи Франкля для уравнения смешанного типа с сингулярным коэффициентом	151
M.Azizov, Q Orifjonova Yuklangan issiqlik tarqalish tenglamasi uchun bir chegaraviy masala	158
<u>Персоналии</u>	
Академик Қ.Ғ.Ғуломов 80 ёшда	166
Правила оформления статьи.....	167

Gyrobifastigium qirralari bo'ylab ikki quvlovchi va bir qochuvchining tutish differensial o'yini

G.A. Vasiyeva

Andijan davlat universiteti, Andijon, 170100, Universitet, 129 (Uzbekistan).

Annotatsiya. Ushbu maqolada biz Gyrobifastigium qirralarida ikki quvlovchi va bitta qochuvchining differensial o'yinini o'rganamiz. O'yinda "asosiy quvlovchi" deb nomlangan bitta quvlovchining maksimal tezligi $\rho_1 = 1$ ga teng. Bu tezlik qochuvchining maksimal tezligi bilan bir xil. Ikkinchi quvlovchining maksimal tezligi esa 0 dan biroz yuqoriroq, ya'ni u doimiy ravishda harakat qilishi uchun yetarli. Quvlovchilardan birining holati qochuvchining holati bilan ustma-ust tushsa, quvlovchilar o'yinda g'alaba qozongan deb hisoblanadi. Natijada o'yinni yakunlash uchun strategiya qurilgan va strategiyadan foydalangan holda tutish mumkinligi isbotlangan hisoblanadi.

Kalit so'zlar: Quvlovchi, qochuvchi, differensial o'yin, strategiya, Gyrobifastigium.

Abstract. In this paper, we study the differential game of two chasers and one escaper on the edges of a Gyrobifastigium. In the game, one pursuer, which is called the "main pursuer" have a maximum speed of $\rho_1 = 1$. It is the same as the evader. However, the maximum speed of the second pursuer is just above 0, which means it is enough for it to move constantly. When the state of one of the pursuers coincides with the state of the evader, the chasers are considered to have won the game. As a result, the strategy to complete the game is constructed, and it is proved that the pursuit is possible by using the strategy.

Key words: Pursuer, evader, differential game, strategy, Gyrobifastigium.

Аннотация. В данной статье мы изучаем дифференциальную игру два преследователя и одного убегающего на рёбрах Гыробифастигиум. В игре один преследователь, которых называют «главный преследователь», имеют максимальные скорости $\rho_1 = 1$. Это то же самое, что и убегающего. Однако максимальная скорость третьего преследователя чуть выше 0, а значит, ей достаточно для постоянного движения. Когда состояние одного из преследователей совпадает с состоянием убегающего, считается, что преследователи выиграли игру. В результате строится стратегия завершения игры и доказывается, что преследование возможно с использованием этой стратегии.

Ключевые слова: Преследователь, убегающий, дифференциальная игра, стратегия, Гыробифастигиум.

1. KIRISH

XX asrning ikkinchi yarmidan boshlab matematik modellash va optimal boshqaruv nazariyasining rivojlanishi natijasida differensial o'yinlar nazariyasi alohida yo'nalish sifatida shakllandi. 1941-1945-yillarda ushbu nazariya uchun asos bo'ladigan turli harbiy sohadagi masalalar o'ranilgan. Differentsial o'yinlarni chuqurroq o'rganish dastlab amerikalik matematik R. R.Ayzeksning XX asrning 50-yillari boshlarida AQSh ning RAND korporatsiyasi buyurtmasi bo'yicha qilingan tadqiqotlarida keltirilgan. R.Ayzeks 1965-yilda o'z tadqiqotlari asosida "Differensial o'yinlar" nomli monografiyasini nashr qildi [10]. Ushbu monografiyada u ko'plab differensial o'yin muammolariga keltirilgan aniq model muammolari tahlilini taqdim etdi va ularning batafsil yechimlarini ko'rsatdi, shuningdek, bir

qator yangi hayotiy muammolarini taklif qildi R. Ayzeksdan keyin L.S. Pontryagin [12], N.N. Krasovskii [11] va A.I. Subbotin, Azamov [1], Satimov [13] kabi tadqiqotchilar tomonidan chuqur natijalar olinib rivojlantirildi.

Yaqin yillardan beri differensial o'yinlar sohasida qizg'in muhokama qilinayotgan eng dolzarb masalalardan biri ko'p quvlovchili differensial o'yinlarga oid masalalardir. Bunday ko'p quvlovchili differensial o'yinlarga bag'ishlangan Chernousko [6], Pshenichnii [14] kabi ishlardan ko'p quvlovchili differensial o'yinlar o'zining dastlabki bosqichlaridayoq tez rivojlanganini kuzatish mumkin. Bu qiziqishning yanada ortib borayotganini quyidagi maqolalardan ham ko'rish mumkin: Ibragimov va Salleh [8], Blagodtskix va

Petrov [5], Azamov v.b. [2], Ibragimov v.b. [9], Ibragimov [7].

Hujumchi va himoya qiluvchilar guruhidan tashqari hujumchi nishonga oladigan maydon ham bo'ladigan differensial o'yinlar [7] maqolada qaralgan. Hujumchining asosiy maqsadi himoya qiluvchilar guruhiga tutilmay o'sha maydonga yetib borishdan iboratdir. Himoya qiluvchilar guruhining maqsadi esa hujumchini u maydonga kiritmaslik, yoki kirishiga imkon boricha halaqit berishdir.

Ushbu [7] maqolada $\mathbb{R}^n$ da ko'p quvlovchili va bir qochuvchili differensial o'yin qaralgan. Quvlovchilarning tezliklari qochuvchining tezligidan kichik. Quvlovchilarning soni nechta (chekli) bo'lishiga qaramay va ular qochuvchini o'rab turganmi yoki yo'qligiga qaramay, qochuvchi uchun quvlovchilarga tutilmay qochib ketish strategiyasi qurilgan.

Differensial o'yinlarning yana bir turi bu graflardagi differensial o'yinlardir. Bunday differensial o'yinlarda o'yinchilar grafning qirralari bo'ylab harakat qilishadi. Bunday o'yinlar, masalan, [3], [4] maqolalarda qaralgan.

Biz ushbu maqola orqali Gyrobifastigiumda quvlash masalasini o'rganamiz. Gyrobifastigium 4 ta muntazam uchburchak, 4 ta kvadratli yoqdan iborat fazoviy shakl. Gyrobifastigiumda 8 ta uch bo'lib, 14 ta bir xil qirraga ega. Bu fazoviy shakl Arximed jismlarining bir turi hisoblanadi (1-rasm). [15] maqola aynan Arximed jismlaridagi differensial o'yinlarga bag'ishlangan birinchi ish hisoblanadi.

2. MATERIAL VA USULLAR

Aytaylik, tomoni bir birlik bo'lgan ixtiyoriy $A_1 \dots A_4 B_2 C_1 C_2$ Gyrobifastigium qirralarida ya'ni G da $P = \{P_1; P_2\}$ quvlovchilar guruhi va bitta E qochuvchi harakatlanayotgan bo'lsin (1-rasm). Biz ularni umumiy o'yinchilar deb ataymiz. O'yinchilar doimo bir-birlarining harakatlari, holatlarini ko'rib, his qilib turadi deb olamiz. $P_i(t), i = 1, 2$, va $E(t)$ lar mos ravishda quvlovchilar va qochuvchining t momentdagi G shakldagi vaziyati hisoblanadi. Jumladan, $t = 0$ momentdagi holat o'yinchilarning boshlang'ich holati deyiladi. Biz quvish

o'yinini o'rganayotganimiz uchun quvlovchilarga o'yinni o'z foydalariga hal qilish uchun strategiyalar quramiz. Qochuvchi esa ixtiyoriy boshqaruvni qo'llab harakatlansin deb olamiz.

Qaralayotgan differensial o'yin quyidagicha berilgan:

$$\begin{aligned} \dot{P}_i(t) &= u_i, & P_i(0) &= P_{i0}, \quad i = 1, 2; \\ \dot{E}(t) &= v, & E(0) &= E_0 \end{aligned}$$

Bu yerda $P_i, E, u_i, v \in G$, $P_i(0) \neq E(0)$ va $|v| \leq \sigma = 1$, $|u_i| \leq \rho_i$, umumiylikka ziyon yetkazmagan holda birinchi quvlovchi P_1 ning tezligini qochuvchining tezligiga, 1 ga teng deb olamiz ($\rho_1 = 1$), ikkinchi quvlovchining tezligi nomigagina mavjud bo'lsa kifoya, ya'ni 0 dan biroz katta bo'lsa, yetadi ($0 < \rho_2 < 1$). P_{10}, P_{20} quvlovchilarning boshlang'ich holatlari va E_0 esa qochuvchining boshlang'ich holati. $B(r)$ orqali radiusi r va markazi R^2 da bo'lgan doiralar to'plamini belgilaylik.

1-Ta'rif. Agar ushbu

$$\dot{P}_i(t) = u_i, P_i(0) = P_{i0}, i = 1, 2;$$

Koshi masalasining $P_i(\cdot)$ yechimi $P_i(t) \in G$, $t \geq 0$, shartni qanoatlantirsa, $u_i(\cdot)$, $u_i: [0, \infty) \rightarrow B(\rho_i)$, funksiya P_i , $i \in \{1, 2\}$ quvlovchilarning joiz boshqaruvi deyiladi. 2-

Ta'rif. Agar ushbu

$$\dot{E}(t) = v, E(0) = E_0$$

Koshi masalasining $E(\cdot)$ yechimi $E(t) \in G$, $t \geq 0$ shartni qanoatlantirsa, $v(\cdot)$, $v: [0, \infty) \rightarrow B(\sigma)$ funksiya E qochuvchining joiz boshqaruvi deyiladi.

Biz quvlash differensial o'yinini qaraymiz. Bunday differensial o'yinlarda quvlovchilar biror strategiyalarni qo'llaydi. Qochuvchi esa ixtiyoriy joiz boshqaruvni qo'llaydi.

Endi quvlovchilarning strategiyalariga ta'rif beramiz.

3-Ta'rif. Agar $u_i = U_i(t, P_1, P_2, E, v)$, $i = 1, 2$ funksiyalar va qochuvchining ixtiyoriy $v = v(t)$ joiz boshqaruvi uchun berilgan Koshi masalasi yagona $P_1(t), P_2(t)$, $E(t)$, $t \geq 0$ yechimga ega bo'lsa, $(t, P_1, P_2, E, v) \rightarrow U_i(t, P_1, P_2, E, v)$, $i = 1, 2$, funksiyalar mos ravishda P_i , $i = 1, 2$ quvlovchilarning strategiyalari deyiladi.

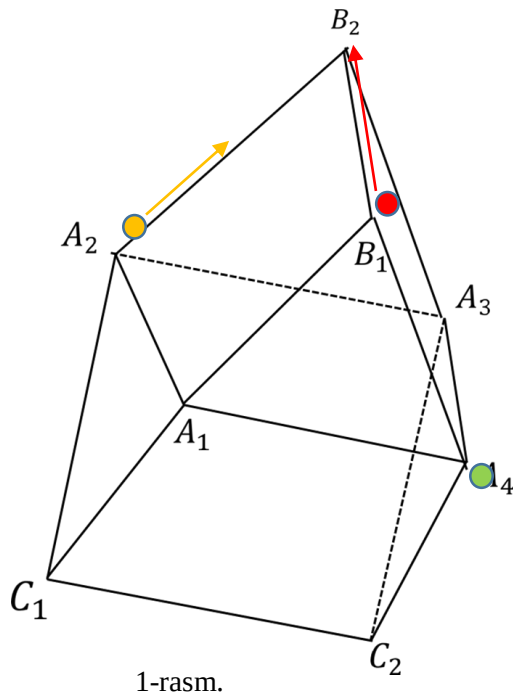
4-Ta'rif. Agar E qochuvchi Gyrobifastigiumning ixtiyoriy qirrasida

joylashgan holda qanday strategiya qo'llamasin, $P = \{P_1; P_2\}$ guruhning shunday strategiyasi va unga binoan shunday $\tau > 0$ soni topilsaki, biror $i = 1, 2$ uchun $P_i(\tau) = E(\tau)$ tenglik o'rinli bo'lsa, u holda o'yinni quvlovchilar guruhi yutadi (ba'zan o'yin tugadi) deyiladi.

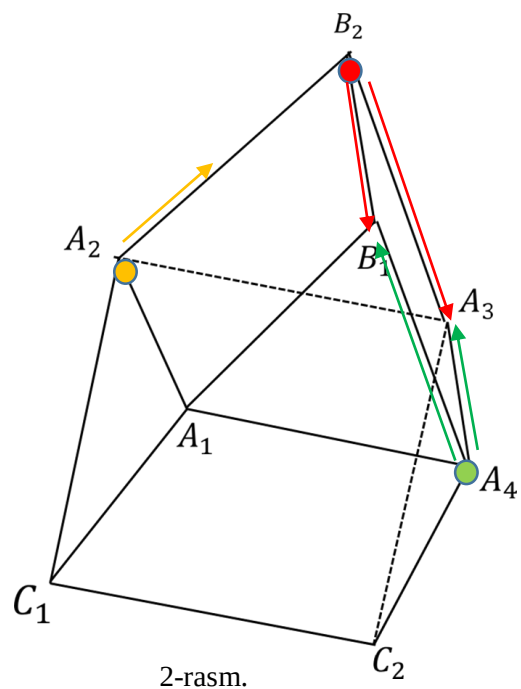
Isbot. Umumiylikka ziyon yetkazmagan holda quvlovchilarni boshlang'ich holatda quyidagicha joylashtiramiz: $P_{10} = A_4$, $P_{20} = A_2$. Qochuvchi esa shaklning ixtiyoriy qirrasida joylashishi mumkin.

3.NATIJALAR VA MUHOKAMA

Teorema. Agar $\sigma=1$ hamda $\rho_1=1$, $\rho_2=\varepsilon$ bo'lsa, o'yinda P quvlovchilar guruhi yutadi.



1-rasm.



2-rasm.

● tezligi 1ga teng quvlovchi, ● tezligi ε ga teng quvlovchi va ● tezligi 1ga teng qochuvchi.

a) qochuvchi B_1 uchda bo'lgan holni ko'rib chiqaylik. Tezligi ε ga teng bo'lgan P_2 quvlovchi B_2 ga harakatlanib uni yurishga majbur qiladi. Bunda quvlovchi A_1 yoki A_4 ga tomon harakatlansa P_1 quvlovchi tomonidan tutiladi. Shu sababli B_2 ga yurishga majbur. Endi qochuvchi B_2 dan qaysi qirraga yurmasin, quvlovchilardan biri tomonidan tutiladi. (2-rasm)

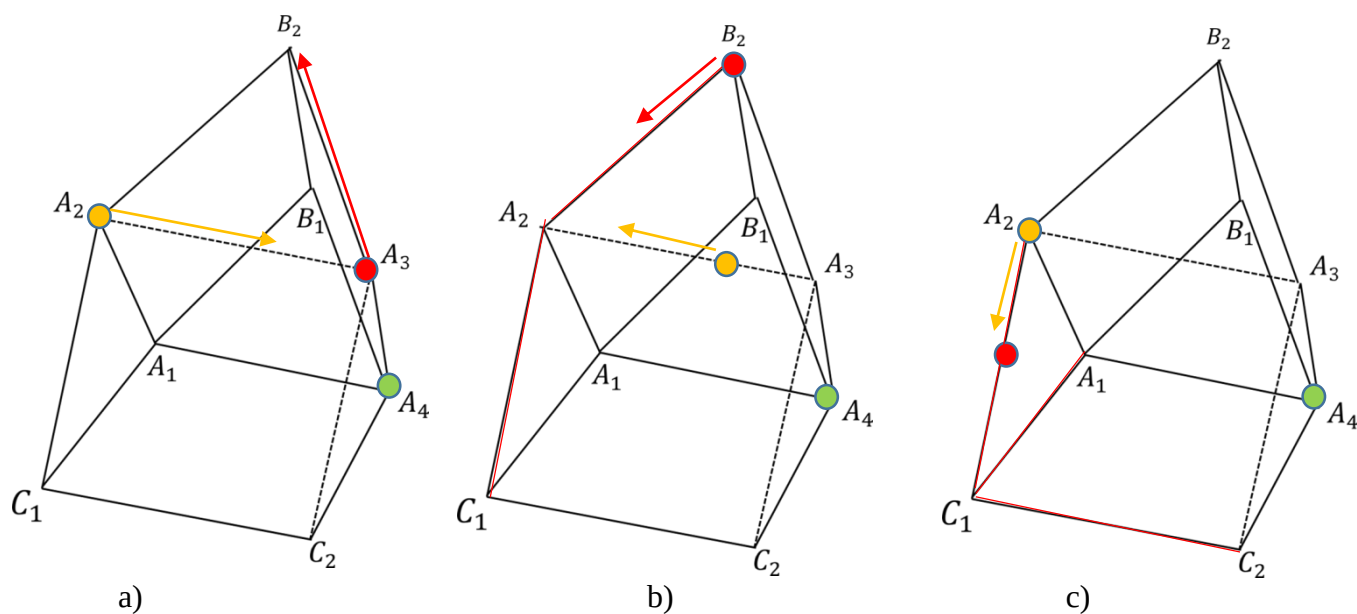
b) qochuvchi B_2 uchda bo'lsin. Yana tezligi ε ga teng bo'lgan P_2 quvlovchi B_2 ga harakatlanib uni yurishga majbur qiladi va qochuvchi qaysi tomonga harakatlanmasin baribir tutiladi. (2-rasm)

c) C_1 va C_2 uchlardagi holatlar mos ravishda B_2 va B_1 uchlariga ekvivalent. Chunki berilgan shakl ikkita bir xil shakldan tashkil topgan.

d) endi qochuvchi A_3 da bo'lgan holatni ko'ramiz. P_2 quvlovchi A_3 tomon yurib, qochuvchini harakatga keltirsin. U faqat B_2 qarab harakatlana oladi (3-a rasm). Qochuvchi B_2 ga yetib kelgach, P_2 quvlovchi ortga, ya'ni A_2 ga qaytadi. Qochuvchi B_1 yoki A_3 ga harakat qilsa, P_1 quvlovchi uni tutib oladi. Demak qochuvchi A_2 tomon harakatlanar ekan (3-b rasm). Endi u faqat $B_2 A_2 C_1$ qirralarda harakat qila oladi. Agar boshqa qirraga tomon harakat qilsa, P_1 quvlovchi tomonidan tutiladi. P_2 quvlovchi A_2 ga yetib

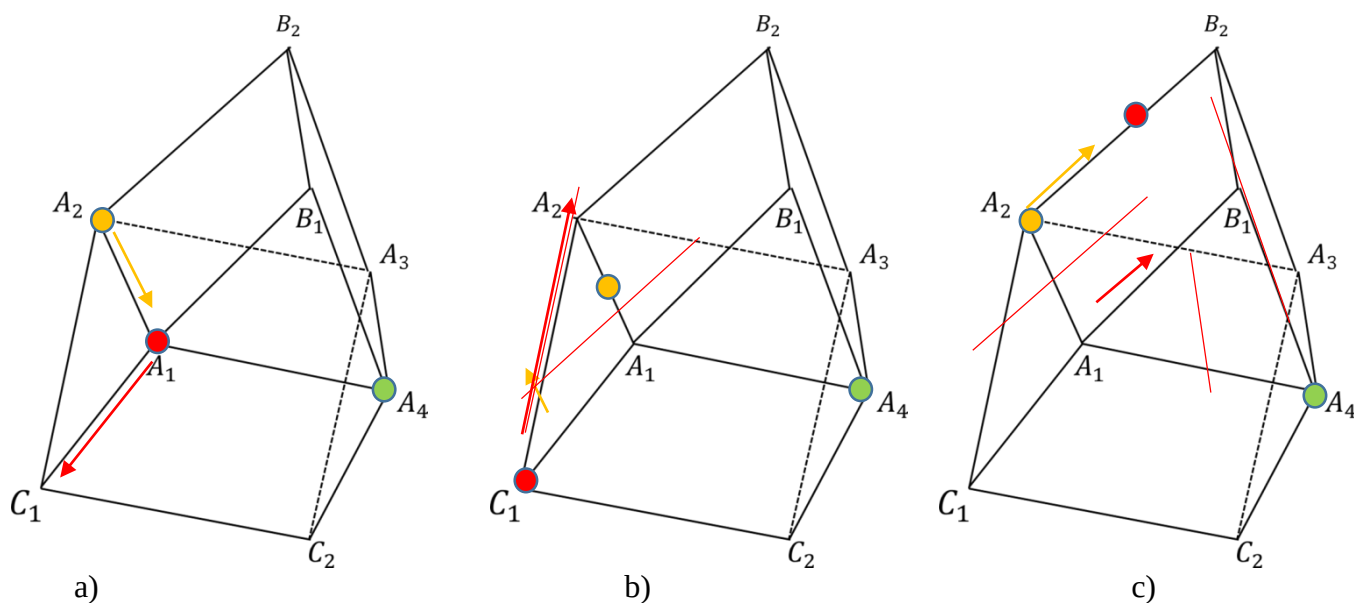
kelgan paytda qochuvchi B_2A_2 yoki A_2C_1 qirrada bo'ladi. Qochuvchi qaysi qirrada bo'lsa, P_2 quvlovchi o' 'sha tomonga harakatlanishni boshlaydi. Natijada

qochuvchining harakati daraxtga aylanib qoladi(3-c rasm). Endi qochuvchi harakat qilsa ham, qilmasa ham tutiladi.



3-rasm.

e) qochuvchi boshlang'ich holatda A_1 da bo'lsin. Bu ham yuqoridagi A_3 dagidek bo'ladi. P_2 quvlovchi u tomonga harakat qiladi va qochuvchi harakatga keladi. Qochuvchi C_1 ga tomon yuradi. Shu uchga yetib kelgach, P_2 quvlovchi o'z o'rniga qayta boshlaydi. Quvlovchi A_2 yetib kelganda, qochuvchining yo'llari to'silib, daraxtga aylanib qoladi. (4-rasm)



4-rasm.

Xuddi shunga o'xshash qochuvchi boshlang'ich holatda shaklning ixtiyoriy qirrasida bo'lganda ham, o'yin shu holatda davom etadi. Ya'ni P_2 quvlovchi harakatlanib,

qochuvchini yurishga majbur qiladi. Qochuvchi C_1 yoki B_2 yetib kelgach, P_2 quvlovchi o'z joyiga, ya'ni A_2 ga qaytib

boradi. P_2 quvlovchi A_2 ga yetib borgunga harakatlanib turadi. P_2 quvlovchi A_2 ga yetib brogan paytdan boshlab qochuvchi qamalga tushadi va o'yinda P quvlovchilar guruhi g'olib bo'ladi.

Teorema isbotlandi.

4. XULOSA

Arximed jismlari orasida hali hal qilinmagan masalalar juda ko'p. So'nggi yillardagi qiziqishlar tufayli bu sohaga katta e'tibor berib kelinmoqda. Jumladan, ushbu maqolada biz Arximed jismlari turiga mansub Gyrobifastigiumning qirralarida 2 quvlovchi

qadar qochuvchi faqat $C_1A_2B_2$ qirralarda va bir qochuvchili differensial o'yin o'rgandik. Bu o'yinda bitta quvlovchi va qochuvchining maksimal tezliklari o'zaro teng. Umumiylikka ziyon yetkazmagan holda, bu tezlik 1 deb olindi. Ikkinchi quvlovchining tezligi esa nomigagina mavjud bo'lsa kifoya, ya'ni 0 dan biroz kattaroq. Bunda biz Gyrobifastigium qirralarida bir qochuvchini tutish uchun eng kamida 2 quvlovchi yetarli ekanini isbotladik. O'yin davomida quvlovchilarga o'yinni yakunlash uchun strategiyalar qurildi. [15] maqola ushbu ishni o'rganishimiz uchun asos bo'ldi.

Adabiyotlar

- [1] Azamov A.A. (1983). On Pontryagin's second method in linear differential games of pursuit. Mat. Sb. (N.S.), 118(160):3(7) (1982), 422–430; Math. USSR-Sb., 46:3, 429–437.
- [2] Azamov A.A., Kuchkarov A.Sh., Holboyev A.G. (2019). The pursuit-evasion game on the 1-skeleton graph of the regular polyhedron. III. Mat. Teor. IgrPril., 11:4 (2019), 5–23.
- [3] Azamov A., Ibaydullaev T. (2020). A pursuit-evasion differential game with slow pursuers on the edge graph of simplexes I. Mathematical Game Theory and Applications. 12(4): 7–23. https://doi.org/10.17076/mgta_2020_4_23
- [4] Azamov A.A., Ibragimov G.I., Ibaydullaev T., Alias I.A. (2021) Optimal number of pursuers in the differential games on the 1-skeleton of orthoplex. Symmetry (Game Theoretical Symmetry Dynamic Processes). 10.3390/sym13112170.
- [5] Blagodatskikh A.I., Petrov N.N. (2019). Simultaneous Multiple Capture of Rigidly Coordinated Evaders. Dynamic Games and Applications (2019) 9:594–613, <https://doi.org/10.1007/s13235-019-00300-8>.
- [6] Chernous'ko F.L. (1976). A Problem of Evasion of Several Pursuers. Prikl. Mat. Mekh., 40(1): 14–24.
- [7] Ibragimov G.I. (2023). Evasion differential game of one evader and many slow pursuers. Dynamic Games and Applications. <https://doi.org/10.1007/s13235-023-00501-2>.
- [8] Ibragimov G.I., Salleh Y. (2012). Simple motion evasion differential game of many pursuers and one evader with integral constraints on control functions of players. Journal of Applied Mathematics, 2012, Article ID 748096, doi:10.1155/2012/748096.
- [9] Ibragimov G.I., Ferrara M., Kuchkarov A.Sh., and Pansera B.A. (2018). Simple motion evasion differential game of many pursuers and evaders with integral constraints. Dynamic Games and Applications. 8: 352–378. <https://doi.org/10.1007/s13235-017-0226-6>.
- [10] Isaacs R. (1965). Differential games. John Wiley & Sons, New York, NY, USA.
- [11] Krasovskii N.N., Subbotin A.I. (1988). Game-Theoretical Control Problems. Springer, New York.
- [12] Pontryagin L.S. (1988). Selected works. Nauka, Moscow.
- [13] Satimov N.Y., Rikhsiev B.B. (2000). Methods of Solving of Evasion Problems in Mathematical Control Theory, Fan, Tashkent, Uzbekistan.
- [14] Pshenichnii B.N. (1976). Simple pursuit by several objects. Cybernetics and Systems Analysis, 12(3):145–146.
- [15] Ibragimov G.I., Xasanjonova N.U. (2023) Kesik tetraedrda ikki quvlovchi va bir qochuvchining differensial o'yini. Scientific Bulletin of NamSU. № 7: 28–36.

“Burilgan ikki tepalikli shakl” qirralarida uch quvlovchi va bir qochuvchining differensial o‘yini

Rayxona Ortiqova Baxodirovna

Annotatsiya. Ushbu maqola burilgan ikki tepalikli shakl qirralari bo‘ylab uch quvlovchilar va bir qochuvchining differensial o‘yiniga bag‘ishlangan. Quvlovchilar va qochuvchining boshqaruv parametrlari geometrik chegaralanishga ega, ya’ni o‘yinchilarning tezliklariga chegaralar qo‘yilgan. O‘yinda quvlovchilarning maksimal tezligi $\rho_1 = 1/3$, $\rho_i = \varepsilon, i = 2, 3$, va qochuvchiniki esa $\sigma = 1$ ga teng. Quvlovchilardan birining biror momentdagi holati qochuvchining huddi shu momentdagi holati bilan ustma-ust tushsa, quvlovchilar o‘yinda g‘alaba qozongan deb ataladi. O‘yinni yakunlash uchun quvlovchilarga strategiyalar qurilgan va strategiyadan foydalangan holda tutish mumkinligi isbotlangan.

Kalit so‘zlar: differensial o‘yin, strategiya, quvlovchi, qochuvchi, burilgan ikki tepalikli shakl.

Аннотация. В данной статье мы рассматривается дифференциальная игра трёх преследователей и одного убегающего на поверхности гиروبифастигия. Управляющие параметры преследователей и убегающего подчиняются геометрическим ограничениям, что означает наличие ограничений по скорости у игроков. В данной игре максимальная скорость преследователей равна: $\rho_1 = \frac{1}{3}$, $\rho_i = \varepsilon, i = 2, 3$, а максимальная скорость убегающего — $\sigma = 1$. Если положение хотя бы одного из преследователей совпадает с положением убегающего, считается, что преследователи выиграли игру. В статье построена стратегия завершения игры для преследователей и доказано, что погоня возможна при использовании данной стратегии.

Ключевое слово: дифференциальная игра, стратегия, преследователь, убегающий, гиروبифастигий.

Abstract. This paper is devoted to study a differential game of three pursuers and an evader on gyrobifastigium. The control parameters of pursuers and an evader are subjected to geometric constraints, which means the players have speed limits. In the game, pursuers have maximum speed of $\rho_1 = \frac{1}{3}$, $\rho_i = \varepsilon, i = 2, 3$, and the evader has maximum speed of $\sigma = 1$. When the state of, at least, one of the pursuers coincides with the one of the evader, it can be said that the pursuers win the game. The strategy for the pursuers to complete the game is constructed and it is proven that the pursuit is possible by using the constructed strategy.

Keywords: differential game, strategy, pursuer, evader, gyrobifastigium.

1. KIRISH

Differensial o‘yin nazariyasi - bu o‘yinchilarning berilgan dinamik sistemaning natijasiga ta’sir ko‘rsatadigan, vaqt o‘tishi bilan tanlov qiladigan ziddiyatli senariylarni o‘rganadigan matematikaning bir tarmog‘idir. Ushbu o‘yinlar doimiy vaqt dinamikasi bilan tavsiflanadi va differentsial tenglamalar bilan ifodalanadi. Klassik misol, quvish-tutish o‘yini bo‘lib, unda bir o‘yinchini, ya’ni quvuvchi boshqa o‘yinchini, qochuvchini, qo‘lga olishni maqsad qiladi, qochuvchi esa o‘z navbatida qo‘lga tushmaslikka intiladi. Bunday o‘yinlarning strategiyalari va natijalari turli sohalarida, jumladan, harbiy taktika, robototexnika va iqtisodiyotda

qo‘llaniladi. Differensial o‘yinlar nazariyasining asoslari Rufus Isaacs [12] tomonidan uning mashhur “**Differential Games: A Mathematical Theory with Applications to Warfare and Pursuit, Control and Optimization**” kitobida qo‘yilgan. Isaacs 1950-yillarda differensial o‘yinlar tushunchasini kiritib, dinamik tizimlarda ziddiyat va hamkorlik muammolarini tahlil qilish hamda hal etish uchun nazariy asos yaratdi. Uning ishlari turli vaziyatlarni, jumladan, quvish-qochish muammolarini o‘rganishga bag‘ishlangan bo‘lib, bu mavzu keyinchalik differensial o‘yinlar tadqiqotining asosiy yo‘nalishlaridan biriga aylangan.

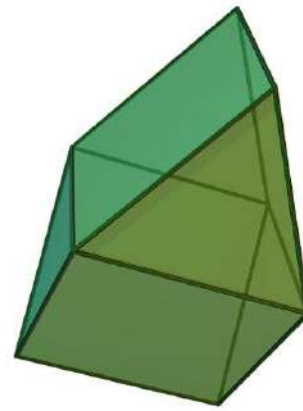
Quvish-qochish o'yinlarida o'yinchilarning dinamikasi ko'pincha ularning harakat va boshqarish imkoniyatlarini tavsiflovchi differensial tenglamalar yordamida aniqlanadi. Quvuvchining maqsadi qochuvchigacha bo'lgan masofani minimallashtirish va shu bilan uni qo'lga olishdir, qochuvchining maqsadi esa qo'lga tushmaslik uchun bu masofani maksimal darajada oshirishga intiladi. Ikkala o'yinchi tomonidan qo'llaniladigan strategiyalar ularning tegishli boshqaruv chegaralanishlarga va o'yin o'tkaziladigan muhitga bog'liq (misol uchun [6-8,10,13-16] adabiyotlarga qarang).

Yana bir qiziqarli sohalardan biri - bu ko'pyoqlar, ko'pburchaklar kabi aniq geometrik shakllar ichida quvish-qochish o'yinlarini tahlil qilishdir. Geometrik shakllarning tuzilishi o'yinchilarning soniga, ularning maksimal tezliklariga, ular qo'llayotgan strategiyalariga va natijalariga sezilarli ta'sir ko'rsatadi ([1-5,9,11] ga qarang).

Burilgan ikki tepalikli shakl bu beshburchak prizmadan uning har bir kvadrat yuziga ikkita teng qirrali kvadrat piramidani biriktirish orqali hosil qilinadi, bu jarayon kengaytma deb ataladi. Bu kvadrat piramidalar prizmaning kvadrat yuzini qoplaydi, shuning uchun hosil bo'lgan ko'pburchakning yuzlari sifatida sakkizta teng yonli uchburchak, uchta kvadrat va ikkita muntazam beshburchak mavjud. Barcha yuzlari muntazam bo'lgan qavariq ko'pburchak "Jonson solid"i deb ataladi va kengaytirilgan beshburchak prizma ular orasida 53-"Jonson solid"i sifatida sanab o'tiladi.

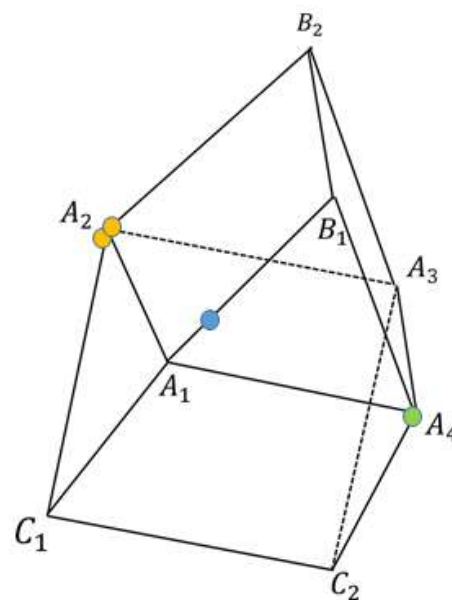
2. MATERIALLAR VA USULLAR

Aytaylik tomoni bir birlik bo'lgan ixtiyoriy $A_1 \dots A_4 B_1, B_2 C_1, C_2$ burilgan ikki tepalikli shaklda (G shakl) $P = \{P_1; P_2; P_3\}$ quvlovchilar guruhi va bitta E qochuvchi harakatlanayotgan bo'lsin (2-rasm). Ular umumiy holatda o'yinchilar



1-rasm. Burilgan ikki tepalikli shakl

deb ataladi. O'yinchilar doimo bir-birlarining harakatlari, holatlarini ko'rib turadi deb olamiz. $P_i(t), i = 1, 2$ va $E(t)$ lar mos ravishda quvlovchilar va qochuvchining t momentdagi ikki



burchakli beshburchak prizmadagi vaziyati hisoblanadi. Shuningdek, $t = 0$ momentdagi holat o'yinchilarning boshlang'ich holati deyiladi. Biz tutish o'yinini o'rganayotganimiz uchun quvlovchilarga o'yinni o'z foydalariga hal qilish uchun strategiyalar quramiz. Qochuvchi esa ixtiyoriy boshqaruvni qo'llab harakatlansin deb olamiz.

Qaralayotgan differensial o'yin quyidagicha berilgan:

$$\begin{cases} \dot{P}_i(t) = u_i, P_i(0) = P_{i0}, i = 1, 2, 3; \\ \dot{E}(t) = v, E(0) = E_0 \end{cases} \quad (1)$$

Bu yerda $P_i, E, u_i, v \in \mathbb{R}^2$, $P_i(0) \neq E(0)$, va $|v| \leq \sigma = 1$, $|u_i| \leq \rho_i = \varepsilon \rho_3 = \frac{1}{3}$, P_{10}, P_{20}, P_{30} quvlovchilarning boshlang'ich holatlari va E_0 esa qochuvchining boshlang'ich holati. $B(r)$ orqali radiusi r va markazi $\mathbb{R}^2$ da bo'lgan doiralar to'plamini belgilaylik.

Ta'rif 1. Agar ushbu

$$\dot{E}(t) = v, E(0) = E_0,$$

Koshi masalasining $E(\cdot)$ yechimi $E(t)$, $t \geq 0$ shartni qanoatlantirsa, $v(\cdot)$, $v: [0, \infty) \rightarrow B(\sigma)$ funksiya E qochuvchining joiz boshqaruvi deyiladi.

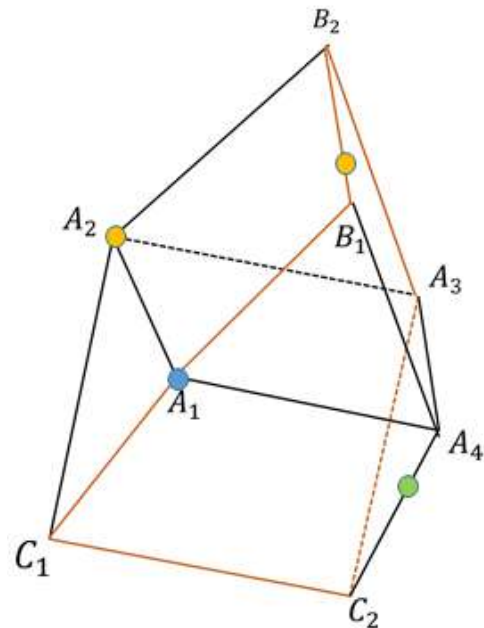
Quvlovchilarning strategiyalariga ta'rif beramiz.

Ta'rif 2. Agar $u_i = U_i(t, x_1, x_2, x_3, y, v)$, $i = 1, 2$, funksiyalar va qochuvchining ixtiyoriy $v = v(t)$ joiz boshqaruvi uchun berilgan (1) Koshi masalasi yagona $P_1(t), P_2(t), P_3(t)$, $E(t)$, $t \geq 0$, yechimga ega bo'lsa, $(t, P_1, P_2, P_3, E, v) \rightarrow U_i(t, P_1, P_2, P_3, E, v)$, $i = 1, 2$ funksiyalar mos ravishda P_i , $i = 1, 2$ quvlovchilarning strategiyalari deyiladi.

Ta'rif 3. Agar E qochuvchi ikki burchakli beshburchak prizmaning ixtiyoriy qirrasida joylashgan holda har qanday joiz boshqaruv qo'llaganda ham, $P = \{P_1; P_2; P_3\}$ quvlovchilar guruhning shunday strategiyasi va unga binoan shunday $\tau > 0$ soni topilsaki, biror $i = 1, 2, 3$ uchun $P_i(\tau) = E(\tau)$ tenglik o'rinli bo'lsa, u holda o'yinni quvlovchilar guruhi yutdi (ba'zan o'yin tugadi) deyiladi.

3. NATIJALAR VA MUHOKAMA

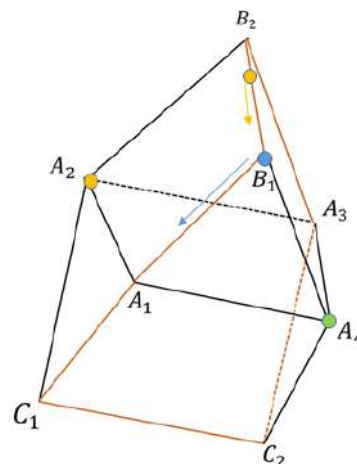
Teorema. Agar $\sigma = 1$ hamda $\rho_1 = 1/3$, $\rho_2 = \rho_3 = \varepsilon$ bo'lsa, o'yinda P quvlovchilar guruhi



yutadi.

Isbot. Umumiylikka ziyon yetkazmagan holda quvlovchilarni ● tezligi $1/3$ ga teng, ● tezligi ε ga teng bo'lgan quvlovchilar. ● tezligi 1 ga teng qochuvchi.

Boshlang'ich holatda quyidagicha joylashtiramiz: $P_1(0) = A_4$, $P_i(0) = A_2$, $i=2, 3$.



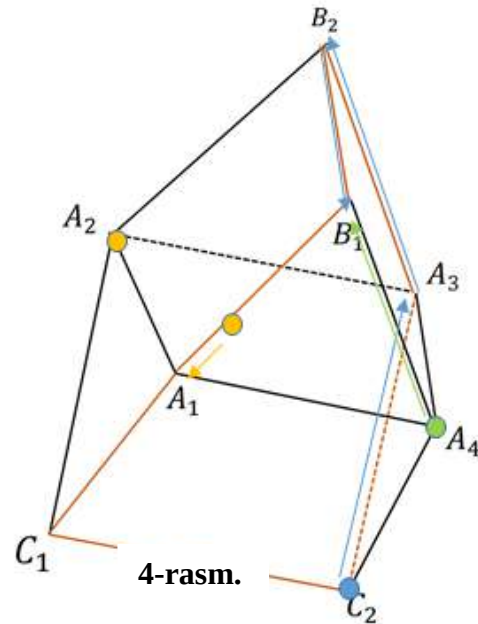
Qochuvchi esa shaklning ixtiyoriy qirrasida joylashishi mumkin. (2-rasm)

a) Qochuvchi A_1B_1 qirrada bo'lgan holni qaraylik. Tezligi ε ga teng bo'lgan quvlovchilardan biri, ya'ni P_2 harakatlanib uni yurishga majbur qilsin. Qochuvchi faqat $A_1C_1C_2A_3B_2B_1$ qirralarda harakat qilishi mumkin. Chunki boshqa qirralar quvlovchilar bilan to'silgan. Qochuvchi boshqa qirralarda yursa, qirra oxirida tutilib qoladi. P_2 ham B_2 uchga yetib kelgach, faqat $A_1C_1C_2A_3B_2B_1$ qirrada va shu yo'nalishda harakatlanadi (3-rasm).

Qochuvchi B_1 ga yetib kelgan paytdan boshlab tezligi $1/3$ ga teng bo'lgan P_1 quvlovchi ham harakatlanishni boshlaydi. Qochuvchi A_1 uchga yetib kelgan vaqtda P_1 quvlovchi A_4C_2 qirraning $1/3$ qismini bosib o'tadi (4-rasm). Endi qochuvchi orqaga, ya'ni B_1 ga qaytsa, P_1 quvlovchi ham o'z o'rniga qaytadi. Bu holat cheksiz davriy davom etsa, P_2 quvlovchi uni A_1B_1 qirrada tutadi. Agar qochuvchi A_1 dan so'ng A_4 tomon yursa, P_1 quvlovchi ham A_4 ga qaytib tutib oladi.

Qochuvchi A_1 dan C_1 ga yetib kelgan paytda P_1 quvlovchi A_4C_2 qirraning $2/3$ qismida joylashadi. Qochuvchi A_2 tomon yursa tutiladi. Agar ortga qaytsa P_1 ham ortga qaytadi va bu holat cheksiz davom etsa, P_2 quvlovchi kelib uni tutib oladi. Shuning uchun qochuvchi C_1 uchdan so'ng C_2 ga harakatlanishga majbur. Bu holatda P_1 quvlovchi ham C_2 ga harakatlanadi va qochuvchini tutadi.

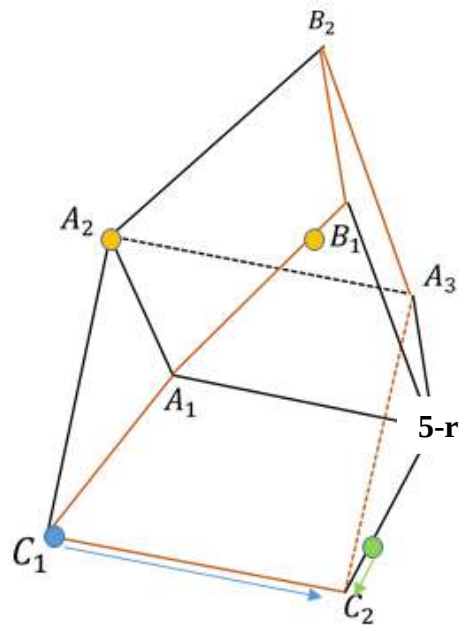
b) Boshqa holatda qochuvchi B_1 uchga yetib kelganda, P_2 quvlovchi qochuvchidan pastda, ya'ni A_1B_1 yoki undan boshqa qirrada joylashgan bo'lsin. Bu holatda ham tezligi ε ga teng P_2 quvlovchi $A_1C_1C_2A_3B_2B_1$ qirralarda qochuvchi harakatiga e'tibor qilmasdan faqat shu yo'nalishda harakat qiladi. Qochuvchi shu qirralarda, lekin unga qarama-qarshi harakatlansa, P_2 quvlovchiga duch kelib qoladi va $A_1C_1C_2A_3B_2B_1$ ning qaysidir qirrasida tutilib qoladi. Shuning uchun u yana P_2 quvlovchi yo'nalishida



4-rasm.

harakatlanishga majbur bo'ladi va B_1 uchdan o'tadi. B_1 uchdan o'tganda esa oldingi strategiya qo'llaniladi.

Bizga berilgan shakl ikkita bir xil shakldan iborat. Shuning uchun qochuvchi xuddi shu yo'nalishda $A_1C_1C_2A_3B_2B_1$



5-rasm.

qirralar bo'ylab harakat qilib, C_2 uchga yetib kelganda ham shu strategiya o'rinli. 1-

holatda qochuvchi C_2 uchda tutilgan bo'lsa, endi B_1 uchda tutiladi (6-rasm).

Demak, qochuvchi shaklning qaysi qirrasini yoki uchida bo'lmasin, P_2 quvlovchi uni harakatlantirib B_1 yoki C_2 uchdan o'tishga majbur qiladi va o'yinda P quvlovchilar guruhi g'olib bo'ladi.

4. XULOSA

Ushbu maqolada biz burilgan ikki tepalikli shakl qirralarida harakatlanuvchi uch quvlovchi va bir qochuvchining differensial o'yinini o'rgandik. Bu o'yinda o'yinchilarning boshqaruv parametrlariga geometrik chegaralanishlar qo'yilgan, ya'ni

o'yinchilarning tezliklariga chegaralar qo'yilgan. O'yinda quvlovchilarning va qochuvchining maksimal tezligi $\rho_1 = 1/3$ $\rho_i = \varepsilon, (i = 2, 3), \sigma = 1$ ga teng. Agar quvlovchilardan birining biror vaqtdagi holati qochuvchining huddi shu vaqtdagi holati bilan ustma-ust tushsa, quvlovchilar guruhi o'yinda g'alaba qozongan deyiladi. O'yinni yakunlash uchun quvlovchilarga strategiyalar qurilgan va strategiyadan foydalangan holda tutish mumkinligi isbotlangan. Umuman olganda ushbu shakldagi o'yinni quvlovchilar foydasi **6-rasm**. qilish uchun 3 quvlovchi yetarli ekan.

Adabiyotlar

1. Azamov A., Ibaydullaev T. (2020). A pursuit-evasion differential game with slow pursuers on the edge graph of simplexes I. Mathematical Game Theory and Applications. 12(4): 7–23. https://doi.org/10.17076/mgta_2020_4_23.
2. Azamov A.A., Ibragimov G.I., Ibaydullaev T., Alias I.A. (2021) Optimal number of pursuers in the differential games on the 1-skeleton of orthoplex. Symmetry (Game Theoretical Symmetry Dynamic Processes). 10.3390/sym13112170.
3. Azamov A.A., Kuchkarov A.Sh., Holboyev A.G. (2015). The pursuit-evasion game on the 1-skeleton graph of the regular polyhedron. I. Mat. Teor. IgrPril., Vol. 7 (2015), 3–15.
4. Azamov A.A., Kuchkarov A.Sh., Holboyev A.G. (2016). The pursuit-evasion game on the 1-skeleton graph of the regular polyhedron. II. Mat. Teor. IgrPril., Vol. 8 (2016), 3–13.
5. Azamov A.A., Kuchkarov A.Sh., Holboyev A.G. (2019). The pursuit-evasion game on the 1-skeleton graph of the regular polyhedron. III. Mat. Teor. IgrPril., 11:4 (2019), 5–23.
6. Ibragimov G.I. (2023). Evasion differential game of one evader and many slow pursuers. Dynamic Games and Applications. <https://doi.org/10.1007/s13235-023-00501-2>.

7. Ibragimov G.I., Ferrara M., Kuchkarov A.Sh., and Pansera B.A. (2018). Simple motion evasion differential game of many pursuers and evaders with integral constraints. Dynamic Games and Applications. 8: 352–378. <https://doi.org/10.1007/s13235-017-0226-6>.
8. Ibragimov G.I., Ferrara M., Ruziboev M., Pansera B.A. (2021). Linear evasion differential game of one evader and several pursuers with integral constraints. International Journal of Game Theory. doi.org/10.1007/s00182-021-00760-6.
9. Ibragimov G.I., Muhammadjonova M.A. (2024). Kuboktaedr qirralari bo'ylab uch quvlovchi va bir qochuvchining tutish differensial o'yini. Buxoro davlat universiteti ilmiy axboroti, Vol. 7 (2024), 50-56.
10. Ibragimov G.I., Salleh Y. (2012). Simple motion evasion differential game of many pursuers and one evader with integral constraints on control functions of players. Journal of Applied Mathematics, 2012, Article ID 748096, doi:10.1155/2012/748096.
11. Ibragimov G.I., Xasanjonova N.U. (2023) Differential game of two pursuers and one evader on a truncated tetrahedron. Scientific Bulletin of NamSU.
12. Isaacs R. (1965). Differential games. John Wiley & Sons, New York, NY, USA.

13. Garcia E., Casbeer D.W., Pachter M. (2020). Optimal strategies for a class of multi-player reach-avoid differential games in 3D space. IEEE Rob. Autom. Lett., 5, 4257–4264, <https://doi.org/10.1109/LRA.2020.2994023>.
14. Kumkov S.S., Le Ménec S., Patsko V.S. Zero-sum pursuit-evasion differential games with many objects: Survey of publications. Dynamic games and applications. 7: 609–633, Doi 10.1007/s13235-016-0209-z.
15. Petrosyan L.A. (1993). Differential Games of Pursuit. World Scientific, Singapore, London.
16. Ramana M.V., and Kothari M. (2017) Pursuit strategy to capture high-speed evaders using multiple pursuers. Journal of Guidance, Control, and Dynamics. 40(1): 139-149, <https://doi.org/10.2514/1.G000584>.

Muallif haqida ma'lumot

Rayxona Ortiqova Baxodirovna - Andijon davlat universiteti matematika va mexanika kafedrasi matematika yo'nalishi 2-bosqich talabasi. E-mail: rayxonabaxodirovna@gmail.com. Tel: +998882147774

n burchakli muntazam prizmada 2 quvlovchi va bir qochuvchining tutish differensial o'yini

Sh. Sh. Maxammadzokirov

Annotatsiya. Ushbu maqolada biz n burchakli muntazam prizma qirralarida ikki quvlovchi va bitta qochuvchining differensial o'yinini o'rganamiz. O'yinda "asosiy quvlovchi" deb ataladigan quvlovchining maksimal tezligi $\rho_1 = 1$ ga teng. Bu qochuvchining maksimal tezligi bilan bir xil. Ikkinchi quvlovchining tezligi esa $\rho_2 = \frac{1}{3}$ ga teng. Quvlovchilardan birining holati qochuvchining holati bilan ustma-ust tushsa, quvlovchilar o'yinda g'alaba qozongan deb hisoblanadi. Natijada, o'yinni yakunlash uchun strategiya qurildi va strategiyadan foydalangan holda tutish mumkinligi isbotlandi.

Kalit so'zlar: Quvlovchi, qochuvchi, differensial o'yin, strategiya.

Аннотация. В данной статье мы изучаем дифференциальную игру с двумя преследователями и одним убегающим на рёбрах правильного n -угольной призме. В игре один из преследователей, называемый "главным преследователем", имеет максимальную скорость $\rho_1 = 1$, что совпадает со скоростью убегающего. Однако максимальная скорость второго преследователя равна $\rho_2 = \frac{1}{3}$. Когда состояние одного из преследователей совпадает с состоянием убегающего, считается, что преследователи выиграли игру. В результате разработана стратегия завершения игры, и доказано, что преследование возможно при использовании этой стратегии.

Ключевые слова: Преследователь, убегающий, дифференциальная игра, стратегия.

Abstract. In this paper, we study the differential game of two chasers and one escaper on the edges of a regular n -angled prism. In the game, a pursuer, which is called the "the main pursuer" have a maximum speed of $\rho_1 = 1$. It is the same as the evader. However, the maximum speed of the second pursuer is $\rho_2 = \frac{1}{3}$. When the state of one of the pursuers coincides with the state of the evader, the chasers are considered to have won the game. As a result, the strategy to complete the game is constructed, and it is proved that the pursuit is possible by using the strategy.

Key words: Pursuer, evader, differential game, strategy.

1. KIRISH

Differensial o'yinlar zamonaviy matematik tadqiqotlarning muhim yo'nalishlaridan biri hisoblanadi. Ular boshqaruv tizimlari, robototexnika, iqtisodiyot, harbiy strategiya va boshqa ko'plab sohalarda amaliy ahamiyatga ega. Bunday o'yinlarda ishtirokchilar o'z maqsadlariga erishish uchun bir-birlariga qarshi harakat qilishadi. Ayniqsa, ta'qib va qochish o'yinlari qiziqish uyg'otib, ularda ta'qibchilar qochuvchini belgilangan hududda qo'lga olishga harakat qilishadi. Ushbu o'yinlar murakkab geometrik shakllar yoki cheklangan muhitlarda o'tkazilganda, ularning tahlili yanada qiyinlashadi.

Yaqin yillarda o'zbek matematik olimlar tomonidan differensial o'yinlarda amalga oshirilayotgan ishlar sohani rivojlantirib kelmoqda. Jumladan, Satimov N.Y. [1], Azamov A.A. [2], Ibragimov G.I. [3], [4], [5], Ibaydullaev T.T. [6], [7], Kuchkarov A.Sh. [8], Holboyev A.G. [9] kabi matematik olimlar sohaning rivojiga ulkan hissa qo'shib kelishmoqda.

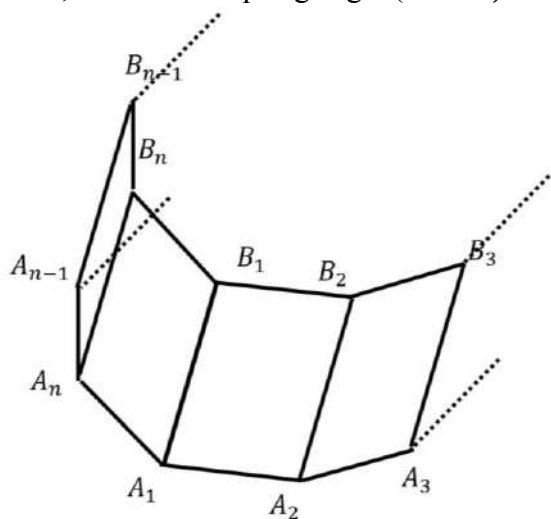
Ushbu maqolada biz ikki quvlovchi va bir qochuvchi ishtirok etadigan differensial ta'qib-qochish o'yinini o'rganamiz. O'yin muntazam n -burchakli prizmaning qirralarida

amalga oshiriladi. Ushbu geometrik konfiguratsiya o'yinni o'zgacha qiziqarli va murakkab qiladi, chunki qirralar bo'ylab harakatlanish graf shaklidagi cheklangan maydonni hosil qiladi. Ta'qibchilardan biri "asosiy ta'qibchi" deb ataladi va uning maksimal tezligi qochuvchining tezligiga teng ($\rho_1 = 1$). Biroq, ikkinchi ta'qibchining maksimal tezligi nisbatan past ($\rho_2 = \frac{1}{3}$), bu esa strategiyani ishlab chiqishda qo'shimcha hisob-kitoblarni talab qiladi.

Bu tadqiqotning ahamiyati shundaki, u cheklangan domenlarda differensial o'yinlarni tahlil qilishga oid yangi usullarni taklif etadi. Ilgari bu yo'nalishda ko'p tadqiqotlar ochiq bo'shliqlar yoki tekis graf shaklidagi muhitlarda olib borilgan bo'lsa, ushbu ish muntazam n -burchakli prizma kabi murakkab shakllarda o'yin strategiyalarini ishlab chiqishga yo'naltirilgan. Bundan tashqari, quvlovchilarning turli imkoniyatlarga ega bo'lishi jamoaviy strategiyalar va rollarni taqsimlashning ahamiyatini yana bir bor namoyon etadi.

Biz ushbu maqola orqali n burchakli muntazam prizmada quvlash masalasini o'rganamiz. n burchakli muntazam prizma (G) shunday fazoviy shaklki, 2 ta muntazam n

burchak, n ta kvadratdan iborat ko'pyoq burchakli muntazam prizmada $2n$ ta uch bo'lib, $3n$ ta bir xil qirraga ega. (1-rasm).



1-rasm

2. MATERIALLAR VA USULLAR

Aytaylik tomoni bir birlik bo'lgan ixtiyoriy $A_1 \dots A_n B_1 \dots B_n$ n burchakli muntazam prizmada (1-rasm) $P = \{P_1; P_2\}$ quvlovchilar guruhi va bitta E qochuvchi harakatlanayotgan bo'lsin. Biz ularni umumiy o'yinchilar deb ataymiz. O'yinchilar doimo bir-birlarining harakatlari, holatlarini ko'rib his qilib turadi deb olamiz. $P_i(t)$, $i = 1, 2$ va $E(t)$ lar mos ravishda quvlovchilar va qochuvchining t momentdagi n burchakli muntazam parallelepipeddagi vaziyati hisoblanadi. Jumladan, $t = 0$ momentdagi holat o'yinchilarning boshlang'ich holati deyiladi. Biz quvlovchi o'yinini o'rganayotganimiz uchun quvlovchilarga o'yinni o'z foydalariga hal qilish uchun strategiyalar quramiz. Qochuvchi esa ixtiyoriy boshqaruvni qo'llab harakatlansin deb olamiz.

Qaralayotgan differensial o'yin quyidagicha berilgan:

$$\begin{aligned} \dot{P}_i(t) &= u_i, & P_i(0) &= P_{i0}, & i &= 1, 2; \\ \dot{E}(t) &= v, & E(0) &= E_0 \end{aligned}$$

Bu yerda $P_i, E, u_i, v \in R^2$, $P_i(0) \neq E(0)$, va $|v| \leq \sigma = 1$, $|u_i| \leq \rho_i$, umumiylikka ziyor yetkazmagan holda, birinchi quvlovchi P_1 ning tezligini qochuvchining tezligiga, 1 ga teng, deb olamiz ($\rho_1 = 1$). Ikkinchi quvlovchi P_2 ning tezligi esa $\rho_2 = \frac{1}{3}$. P_{10}, P_{20} quvlovchilarning boshlang'ich holatlari va E_0

esa qochuvchining boshlang'ich holati. $B(r)$ orqali radiusi r va markazi R^2 da bo'lgan doiralar to'plamini belgilaylik.

2.1-Ta'rif. Agar ushbu

$$\dot{P}_i(t) = u_i, P_i(0) = P_{i0}, i = 1, 2;$$

Koshi masalasining $P_i(\cdot)$ yechimi $P_i(t) \in G$, $t \geq 0$, shartni qanoatlantirsa, $u_i(\cdot)$, $u_i: [0, \infty) \rightarrow B(\rho_i)$, funksiya P_i , $i \in \{1, 2\}$, quvlovchilarning joiz boshqaruvi deyiladi.

2.2-Ta'rif. Agar ushbu

$$\dot{E}(t) = v, E(0) = E_0,$$

Koshi masalasining $E(\cdot)$ yechimi $E(t) \in G$, $t \geq 0$, shartni qanoatlantirsa, $v(\cdot)$, $v: [0, \infty) \rightarrow B(\sigma)$ funksiya E qochuvchining joiz boshqaruvi deyiladi.

Biz quvlovchi differensial o'yinini qaraymiz. Bunday differensial o'yinlarda quvlovchilar biror strategiyalarni qo'llaydi, qochuvchi esa ixtiyoriy joiz boshqaruvni qo'llaydi.

Endi quvlovchilarning strategiyalariga ta'rif beramiz.

2.3-Ta'rif. Agar $u_i = U_i(t, x_1, x_2, y, v)$, $i = 1, 2, 3$, funksiyalar va qochuvchining ixtiyoriy $v = v(t)$ joiz boshqaruvi uchun berilgan Koshi masalasi yagona $P_1(t), P_2(t), E(t)$, $t \geq 0$, yechimga ega bo'lsa, $(t, P_1, P_2, E, v) \rightarrow U_i(t, P_1, P_2, E, v)$, $i = 1, 2, 3$, funksiyalar mos ravishda P_i $i = 1, 2$ quvlovchilarning strategiyalari deyiladi.

2.4-Ta'rif. Agar E qochuvchi kuboktaedrning ixtiyoriy qirrasida joylashgan holda qanday strategiya qo'llamasin, $P = \{P_1; P_2\}$ guruhning shunday strategiyasi va unga binoan shunday $\tau > 0$ soni topilsaki, biror $i = 1, 2$ uchun $P_i(\tau) = E(\tau)$ tenglik o'rinli bo'lsa, u holda o'yinni quvlovchilar guruhi yutdi (ba'zan o'yin tugadi) deyiladi.

Quyidagi teorema ushbu maqolaning asosiy natijasi hisoblanadi.

3. NATIJALAR VA MUHOKAMA

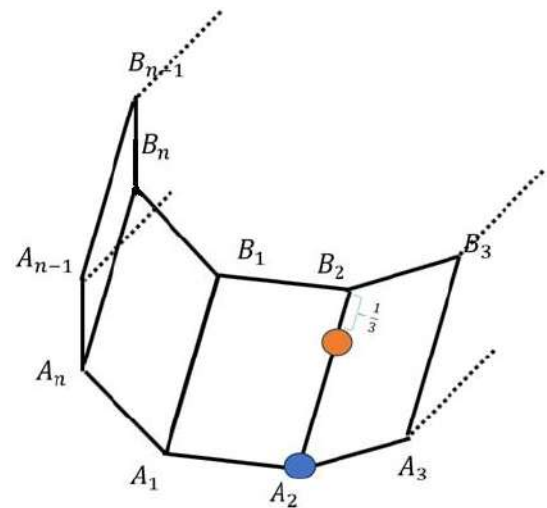
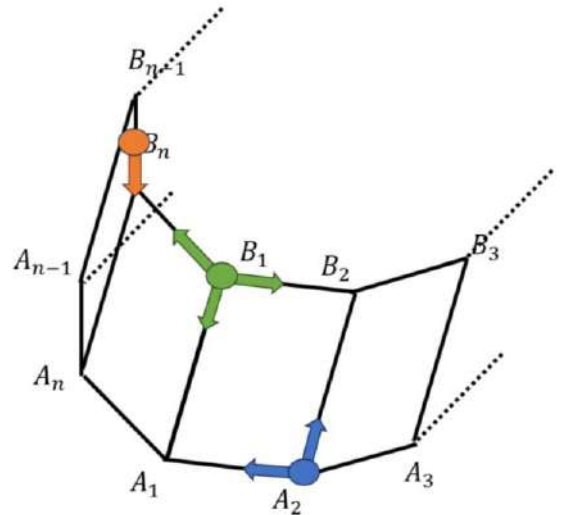
3.1-Teorema. Agar $\sigma = 1$ hamda $\rho_1 = 1$, $\rho_2 = \frac{1}{3}$ bo'lsa, P quvlovchilar guruhi n

burchakli muntazam prizma qirralarida o'yinni yutadi.

Isbot.Tutilish holi. Avvalambor o'yining tutilish holini ko'rib olamiz. Aytaylik, birinchi quvlovchi P_1 biror $A_i (i = \overline{1, n})$ uchda ($B_i (i = \overline{1, n})$ uch uchun ham huddi shunday strategiya tuziladi) uchda bo'lsin (ikkinchi quvlovchi P_2 ning boshlang'ich vaziyati ahamiyatsiz). Qochuvchi esa shu paytda $B_{i-1}B_i (i = \overline{1, n}, \text{ agar } i = 1 \text{ bo'lsa } i - 1 \text{ ni } n \text{ deb olamiz})$ yoki $B_iB_{i+1} (i = \overline{1, n}, \text{ agar } i = n \text{ bo'lsa } i + 1 \text{ ni } 1 \text{ deb olamiz})$ qirrada bo'lsin. Bu vaziyatda P_1 quvlovchi qochuvchi B_{i-1} yoki B_{i+1} uchga kelmaguncha harakatlanmaydi. P_2 quvlovchi B_n burchakka kelib, qochuvchi tomon harakatlanadi va natijada qochuvchi bu uchlardan biridan o'tishga majbur bo'ladi. Aytaylik B_{i-1} uchdan o'tsin (B_{i+1} uch uchun ham huddi shu strategiya o'rinli). Shu vaziyatdan boshlab agar qochuvchi B_i uch tomon harakat qilsa P_1 quvlovchi, qochuvchi B_i uchdan qancha masofada turgan bo'lsa, B_i uch bilan shuncha masofani saqlab harakat qiladi. Natijada qochuvchi A_i uchga kela olmaydi. Agar qochuvchi B_{i+2} uch tomon harakat qilsa, P_1 quvlovchi, qochuvchi B_{i+2} uchdan qancha masofada bo'lsa, A_{i+1} uch bilan shuncha masofani saqlaydi. Qochuvchi B_{i+2} uchga yetib kelsa, shu paytdan boshlab P_1 quvlovchi yuqoridagidek harakat qiladi va natijada qochuvchi B_{i+1} uchga kela olmaydi. P_1 quvlovchining bunday harakat qilishi natijasida qochuvchi B_n burchakda faqat bir tomonga harakat qilgan holda harakatlanishi mumkin. P_2 quvlovchi esa qochuvchi harakatlanishi mumkin bo'lgan tomonga teskari ravishda harakatlanadi va natijada uni Keyingi ko'rilyotgan hollarda „Tutilish holi“ ga olib kelinganda tutildi deb hisoblaymiz.

Aytaylik, quvlovchilar n burchakli muntazam prizma G ning $A_i (i = \overline{1, n})$ uchida turgan bo'lsin. Aks holda biz P_1 va P_2 larni $A_i (i = \overline{1, n})$ uchga keltirib olamiz. P_2 quvlovchi A_iB_i qirrada B_i uchdan $\frac{1}{3}$ masofa uzoqlikdagi nuqtaga keladi. Bu paytda qochuvchi G ning ixtiyoriy nuqtasida

joylashgan bo'lsin (3-rasm). tutadi(2-rasm).



3-rasm

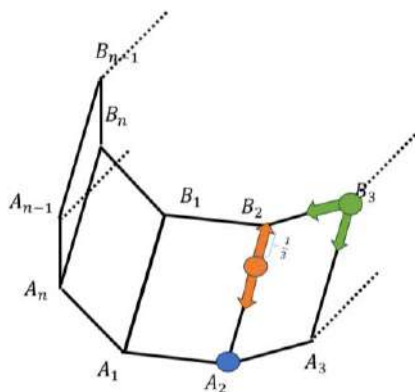
Shu paytdan boshlab quyidagi hollarni ko'rib chiqamiz:

1-hol. Aytaylik qochuvchi $B_{i-1}B_i (i = \overline{1, n}, \text{ agar } i = 1 \text{ bo'lsa } i - 1 \text{ ni } n \text{ deb olamiz})$ yoki $B_iB_{i+1} (i = \overline{1, n}, \text{ agar } i = n \text{ bo'lsa } i + 1 \text{ ni } 1 \text{ deb olamiz})$ qirrada bo'lsin. U holda bu hol „Tutilish holi“ga aynan mos.

2-hol. Qochuvchi G ning $B_{i-1}B_i (i = \overline{1, n}, \text{ agar } i = 1 \text{ bo'lsa } i - 1 \text{ ni } n \text{ deb olamiz})$ yoki $B_iB_{i+1} (i = \overline{1, n}, \text{ agar } i = n \text{ bo'lsa } i + 1 \text{ ni } 1 \text{ deb olamiz})$ qirrasidan tashqari ixtiyoriy nuqtasida bo'lsin. Bu holda P_1 quvlovchi qochuvchining A_n burchakdagi soyasi tomon eng qisqa yo'l bilan harakatlanadi.

Ta'rif. Qochuvchining soyasi deyilganda, u turgan nuqtadan yon qirralarga parallel to'g'ri chiziqli o'tkizilganda, bu to'g'ri chiziqli A va B n burchaklar bilan kesishish nuqtalariga aytiladi.

Bu paytda P_2 quvlovchining vazifasi esa qochuvchini A_i va B_i uchlarga keltirmaslikdan iborat. Buning uchun u agar qochuvchi B_{i+1} (B_{i-1} uch uchun ham shunday) uchga kelib B_i uch tomon harakatlansa, qochuvchi B_i uchdan qancha masofada bo'lsa, u B_i uchdan shu masofaning $\frac{1}{3}$ qismi bilan teng masofani saqlab harakat qiladi (4-rasm).



Agar qochuvchi B_{i+1} (B_{i-1} uch uchun ham shunday) uchga kelib A_{i+1} uch tomon harakatlansa P_2 quvlovchi ham A_i uchdan $\frac{1}{3}$ masofa uzoqlikdagi nuqta tomon harakatlanadi. Agar qochuvchi A_{i+1} uchga kelib A_i uch tomon harakatlansa, qochuvchi A_i uchdan qancha masofada bo'lsa, P_2 quvlovchi A_i uchdan shu masofaning $\frac{1}{3}$ qismi bilan teng masofani saqlab harakat qiladi. Agar qochuvchi A_{i+2} uchga kelib A_i uch tomon harakatlansa ham P_2 quvlovchi huddi shunday harakat qiladi. P_1 quvlovchining harakati tufayli qochuvchi A_i uchga A_{i+2} uchdan o'tmasdan kela olmaydi. Shunday ekan P_2 quvlovchining bunday harakati natijasida qochuvchi A_i va B_i uchlarga kela olmaydi. Natijada qochuvchi A va B n burchaklarning birortasi to'liq sikl bo'ylab aylanib chiqolmaydi. Natijada P_1 quvlovchi uzog'i bilan A n burchakni bir marta aylanib chiqib qochuvchining A n burchakdagi soyasini tutadi. Soyasini tutganda esa o'yin „Tutilish holi“ga keladi.

4. XULOSA

Ushbu maqolada n burchakli muntazam prizmaning qirralarida 2 quvlovchi va bir qochuvchili differensial o'yin o'rganildi.

n burchakli muntazam prizma qirralarida bir qochuvchini tutish uchun eng kamida 2 quvlovchi yetarli ekan. O'yin davomida quvlovchilarga o'yinni yakunlash uchun strategiyalar qurildi.

Adabiyotlar

1. Satimov N.Y., Rikhsiev B.B. (2000). Methods of Solving of Evasion Problems in Mathematical Control Theory, Fan, Tashkent, Uzbekistan.
2. Azamov A.A. (1983). On Pontryagin's second method in linear differential games of pursuit. Mat. Sb. (N.S.), 118(160):3(7) (1982), 422–430; Math. USSR-Sb., 46:3, 429–437.
3. Azamov A., Ibaydullaev T. (2020). A pursuit-evasion differential game with slow pursuers on the edge graph of simplexes I. Mathematical Game Theory and Applications. 12(4): 7–23.
4. Ibragimov G.I., Ferrara M., Ruziboev M., Pansera B.A. (2021). Linear evasion differential game of one evader and several pursuers with integral constraints. International Journal of Game Theory. doi.org/10.1007/s00182-021-00760-6
5. Ibragimov G.I., Salleh Y. (2012). Simple motion evasion differential game of many pursuers and one evader with integral constraints on control functions of players. Journal of Applied Mathematics, 2012, Article ID 748096, doi:10.1155/2012/748096.
6. Ibragimov G.I. (2023). Evasion differential game of one evader and many slow pursuers. Dynamic Games and Applications. <https://doi.org/10.1007/s13235-023-00501-2>.
7. Azamov A., Ibaydullaev T. (2020). A pursuit-evasion differential game with slow pursuers on the edge graph of simplexes I. Mathematical Game Theory and Applications. 12(4): 7–23. https://doi.org/10.17076/mgta_2020_4_23
8. Azamov A.A., Ibragimov G.I., Ibaydullaev T., Alias I.A. (2021) Optimal number of pursuers in the differential games on the 1-skeleton of orthoplex. Symmetry (Game Theoretical Symmetry Dynamic Processes). 10.3390/sym13112170.
9. Ibragimov G.I., Ferrara M., Kuchkarov A.Sh., and Pansera B.A. (2018). Simple motion evasion differential game of many pursuers and evaders with integral constraints. Dynamic Games and Applications. 8: 352–378. <https://doi.org/10.1007/s13235-017-0226-6>.
10. Azamov A.A., Kuchkarov A.Sh., Holboyev A.G. (2019). The pursuit-evasion game on the 1-

skeleton graph of the regular polyhedron. III. Mat.
Teor. IgrPril., 11:4 (2019), 5–23

Muallif haqida ma'lumot

Maxammadzokirov Shoxrux Shuxratbek
o'g'li - Andijon davlat universiteti matematika va
mexanika kafedrası matematika yo'nalishi 2-
bosqich talabasi. Email:
matematikshohrux@gmail.com. Telefon:
+998945628018.

Estimation of accuracy of the numerical solution of Cauchy problem for quadratic dynamical systems using Runge-Kutta methods

Odiljon S.Akhmedov, Dilshoda B.Muzaffarova

Andijan State University, Andijan, 170100, str. University, 129 (Uzbekistan).

Аннотация. В статье, с использованием методов Рунге-Кутты, применяемых для численного решения дифференциальных уравнений, найдена оценка разности между приближенным и численным решениями квадратичной системы.

Ключевые слова: динамическая система, численные методы, метод Рунге-Кутты, метод DN-слежения, билинейное отображение.

Abstract. In the article, using the Runge-Kutta methods, which is used to solve differential equations numerically, an estimate for the difference between the approximate and numerical solutions of the quadratic system is found.

Keywords: dynamical system, numerical methods, Runge-Kutta method, DN-tracking method, bilinear mapping.

Annotatsiya. Maqolada differensial tenglamalarni sonli yechishda qo'llaniladigan Runge-Kutta usullaridan foydalanib, kvadratik dinamik sistemalar uchun taqribiy va sonli yechimlar o'rtasidagi farqning bahosi topilgan.

Kalit so'zlar: dinamik sistema, sonli usullar, Runge-Kutta usuli, DN-kuzatuv usuli, bilinear akslantirish.

1. INTRODUCTION

In the work [1] A.Azamov proposed a new method of DN-tracking, based on the idea of using a numerical method to establish that a specific trajectory of a dynamical system has one or another property. This method is based on an estimate of the difference between exact and discrete solutions, as well as between discrete and numerical solutions for dynamical systems.

Estimation of the difference between exact and approximate solutions for quadratic systems was obtained in the article [2]. In this paper an estimate is derived for the difference between the approximate and numerical solutions for quadratic systems.

2. MATERIALS AND METHODS

Let us consider a quadratic system of the form

$$\dot{z} = f(z), \quad z(0) = z_0, \quad z \in \mathbb{R}^d \quad (1)$$

where $z = (z_1, z_2, \dots, z_d)$,

$$f_k(z) = c^k + \sum_{i=1}^d a_i^k z_i + \sum_{i=1}^d \sum_{j \geq i}^d b_{ij}^k z_i z_j,$$

It's known that the sizes of computers memory are limited meaning that binary cells are fixed and the system of representable real numbers

$$c^k, a_i^k, b_{ij}^k \in \mathbb{R}, \quad i, j, k = \overline{1, d}.$$

Assumption. Solution $z(t)$ of the system (1) exists on the given segment $[0, T]$ and does not leave the convex compact region K .

Let us use the general Runge-Kutta scheme of the second order of accuracy to solve (1) numerically with integrating step $\square = \frac{T}{N}$ [3-5]:

$$z_{n+1} = z_n + p_1 k_1(z, \square) + p_2 k_2(z, \square) \quad (2)$$

where $k_1(z, \square) = \square f(z)$,

$$k_2(z, \square) = \square f(z + \beta_{21} k_1(z, \square)),$$

$$p_1 + p_2 = 1, 2\beta_{21}p_2 = 1$$

N – is the number of division of the segment $[0, T]$.

Definition 1. The sequence of real numbers $z_1, z_2, \dots, z_N$ is called a discrete solution of the Cauchy problem (1).

are bounded. Therefore, in real calculations due to rounding of results (on computer) instead of discrete solution $\{z_n\}$, another sequence of vectors ζ_n is obtained.

Definition 2. The sequence of real numbers $\zeta_1, \zeta_2, \dots, \zeta_N$ is called a numerical solution of the Cauchy problem (1).

Definition 3. $\mathbf{rd}(x)$ (abbreviation for *rounding*) denotes the result of rounding a real number x accurate to decimal places k signs after decimal separator, i.e. $\mathbf{rd}(x) = 10^{-k}[10^k x + 0.5]$, where $[\cdot]$ – integer part sign, k – a fixed positive integer.

By definition, it is easy to see that

$$|x - \mathbf{rd}(x)| \leq \Delta = 0.5 \cdot 10^{-k},$$

For the vector $\eta \in \square^d$, we set $\mathbf{rd}(\eta) = (\mathbf{rd}(\eta_1), \mathbf{rd}(\eta_2), \dots, \mathbf{rd}(\eta_d))$.

Introduce notations

$$\max_{z \in K} |k_2(z, \square)| \leq m_0 \square + \beta_{21} m_0 m_1 \square^2 + \frac{\beta_{21}^2}{2} m_0^2 m_2 \square^3 = I_2(\square).$$

3. RESULTS AND DISCUSSION

Theorem. Let Assumption and conditions $z_n \in K, \zeta_n \in K$ hold for all $n = \overline{0, N}$. Then

$$|z_n - \zeta_n| < \frac{e^{LT} - 1}{L \square} \Delta_{rd}, \quad n = \overline{0, N}.$$

where $L = p_1 L_1 + p_2 L_2$; $L_1 = m_1$; $L_2 = (m_1 + \beta_{21} m_2 I_1(\square))(1 + \beta_{21} \square L_1)$, Δ_{rd} – rounding error at one step.

Proof. Consider the operator

$$R[z] = z + p_1 k_1(z, \square) + p_2 k_2(z, \square).$$

Then the scheme takes the form $z_{n+1} = R[z_n]$.

Let's put $\bar{z}_{n+1} = R[\zeta_n]$, then

$$R[x] - R[y] = x - y + p_1(k_1(x, h) - k_1(y, h)) + p_2(k_2(x, h) - k_2(y, h))$$

Applying Adamar's lemma [4]

$$|k_1(x, h) - k_1(y, h)| \leq h \left\| \int_0^1 f'(\sigma_x^y) d\sigma \right\| |x - y| \leq m_1 h |x - y|$$

where $\sigma_x^y = \sigma x + (1 - \sigma)y$.

For the vector function k_2 :

$$\begin{aligned} |k_2(x, h) - k_2(y, h)| &\leq h \left\| \int_0^1 f'(\sigma_x^y + \beta_{21} \sigma k_1(x, h) + \beta_{21} (1 - \sigma) k_1(y, h)) d\sigma \right\| \\ &\times |x - y + \beta_{21} (k_1(x, h) - k_1(y, h))|. \end{aligned}$$

Furthermore, due to the quadraticity of f we obtain inequality

$$\|f'(\sigma_x^y + \beta_{21} \sigma k_1(x, h) + \beta_{21} (1 - \sigma) k_1(y, h))\| \leq \|f'(\sigma_x^y)\| + \|f''(\sigma_x^y)(\beta_{21} \sigma k_1(x, h) + \beta_{21} (1 - \sigma) k_1(y, h))\| \leq m_1 + \beta_{21} m_2 I_1(h).$$

Thus,

$$|k_2(x, h) - k_2(y, h)| \leq h (m_1 + \beta_{21} m_2 I_1(h))(1 + \beta_{21} h L_1) |x - y| = L_2 h |x - y|.$$

So

$$\begin{aligned} m_0 &= \max_{z \in K} |f(z)|; \quad m_1 = \max_{z \in K} |f'(z)|; \\ m_2 &= \max_{z \in K} \|f''(z)\| \end{aligned}$$

where $|\cdot|, \|\cdot\|, \|\cdot\|$ – Euclidean norms of the vector $f(z)$, linear mapping $f'(z)$ and bilinear mapping $f''(z)$, respectively.

Note that the maximum values of vector functions $k_1(z, \square), k_2(z, \square)$ are easy to estimate in terms of m_0, m_1, m_2 , namely

$$\max_{z \in K} |k_1(z, \square)| \leq m_0 \square = I_1(\square),$$

$$|R[x] - R[y]| \leq (1 + Lh)|x - y|,$$

where $L = p_1 L_1 + p_2 L_2$.

Obviously, the vector ζ , which is obtained by computer the vector $\mathbf{rd}(\zeta)$ coincides with its argument, i.e. $\zeta = \mathbf{rd}(\zeta)$. Let us estimate the error in calculating $f(\zeta)$.

Using the definition of $\mathbf{rd}()$, we have

$$|a_i^k \zeta_i - \mathbf{rd}(a_i^k \zeta_i)| \leq \Delta, \quad |\zeta_i \zeta_j - \mathbf{rd}(\zeta_i \zeta_j)| \leq \Delta,$$

Multiplying both sides of the last inequality by $|b_{ij}^k|$, we get

$$|b_{ij}^k \zeta_i \zeta_j - b_{ij}^k \mathbf{rd}(\zeta_i \zeta_j)| \leq |b_{ij}^k| \Delta, \quad (3)$$

By definition of $\mathbf{rd}()$

$$|b_{ij}^k \mathbf{rd}(\zeta_i \zeta_j) - \mathbf{rd}(b_{ij}^k \mathbf{rd}(\zeta_i \zeta_j))| \leq \Delta, \quad (4)$$

By (3) and (4)

$$|b_{ij}^k \zeta_i \zeta_j - \mathbf{rd}(b_{ij}^k \mathbf{rd}(\zeta_i \zeta_j))| \leq |b_{ij}^k| \Delta + \Delta,$$

All obtained inequalities imply

$$\left| \sum_i a_i^k \zeta_i - \sum_i \mathbf{rd}(a_i^k \zeta_i) \right| \leq \sum_i \Delta = d\Delta,$$

$$\left| \sum_i \sum_{j \geq i} b_{ij}^k \zeta_i \zeta_j - \sum_i \sum_{j \geq i} \mathbf{rd}(b_{ij}^k \mathbf{rd}(\zeta_i \zeta_j)) \right| \leq \sum_i \sum_{j \geq i} (|b_{ij}^k| + 1) \Delta$$

Therefore,

$$\begin{aligned} \delta_k &= \left| f_k(\zeta) - (c^k + \sum_i \mathbf{rd}(a_i^k \zeta_i) + \sum_i \sum_{j \geq i} \mathbf{rd}(b_{ij}^k \mathbf{rd}(\zeta_i \zeta_j))) \right| \\ &\leq \left(d + \sum_i \sum_{j \geq i} (|b_{ij}^k| + 1) \right) \Delta. \end{aligned}$$

So, the error of $f(\zeta)$ is estimated as follows

$$|\delta_\Sigma| \leq \sqrt{\sum_k \delta_k^2}$$

where

$$\begin{aligned} \delta_\Sigma &= f(\zeta) - \mathbf{Rd}(f(\zeta)), \quad \mathbf{Rd} f(\zeta) = (\mathbf{Rd} f_1(\zeta), \mathbf{Rd} f_2(\zeta), \dots, \mathbf{Rd} f_d(\zeta)), \\ \mathbf{Rd}(f_k(\zeta)) &= \sum_i \mathbf{rd}(a_i^k \zeta_i) + \sum_i \sum_{j \geq 1} \mathbf{rd}(b_{ij}^k \mathbf{rd}(\zeta_i \zeta_k)) \end{aligned}$$

This implies,

$$|\square f(\zeta) - \square \mathbf{Rd}(f(\zeta))| \leq \square \sqrt{\sum_k \delta_k^2} \leq \square \sqrt{\sum_k \left(d + \sum_i \sum_{j \geq i} (|b_{ij}^k| + 1) \right)^2} \Delta.$$

and,

$$|\square \mathbf{Rd}(f(\zeta)) - \mathbf{rd}(\square \mathbf{Rd}(f(\zeta)))| \leq \Delta \sqrt{\sum_k 1^2} = \Delta \sqrt{d}.$$

Finally, for the error in calculating the value of $k_1(\zeta, \square)$ we obtain the estimate

$$|\delta_{k_1}| \leq \left(\square \sqrt{\sum_k \left(d + \sum_i \sum_{j \geq i} (|b_{ij}^k| + 1) \right)^2} + \sqrt{d} \right) \Delta = \Delta_1 \quad (5)$$

where $\delta_{k_1} = \square f(\zeta) - \mathbf{rd}(\square \mathbf{Rd}(f(\zeta)))$

Now, let's proceed to deriving an estimate for the calculation error of function k_2 . For this purpose, we first determine $f(\zeta + \beta_{21}k_1)$. By (5) we have

$$k_1(\zeta, \square) = \tilde{k}_1(\zeta, \square) + \delta_{k_1}$$

where $\tilde{k}_1(\zeta, \square) = \mathbf{rd}(\square \mathbf{Rd}(f(\zeta)))$.

Therefore

$$f(\zeta + \beta_{21}k_1(\zeta, \square)) = f(\zeta + \beta_{21}\tilde{k}_1(\zeta, \square) + \beta_{21}\delta_{k_1}).$$

$$\text{Let } \delta_{\tilde{k}_1} = \beta_{21}\tilde{k}_1(\zeta, \square) - \mathbf{rd}(\beta_{21}\tilde{k}_1(\zeta, \square))$$

then

$$f(\zeta + \beta_{21}\tilde{k}_1(\zeta, \square) + \beta_{21}\delta_{k_1}) = f(\zeta + \mathbf{rd}(\beta_{21}\tilde{k}_1(\zeta, \square)) + \delta_{\tilde{k}_1} + \beta_{21}\delta_{k_1}).$$

Expanding the expression into a Taylor series at the point $\zeta_r = \zeta + \mathbf{rd}(\beta_{21}\tilde{k}_1(\zeta, \square))$, we obtain

$$f(\zeta_r + \delta_{\tilde{k}_1} + \beta_{21}\delta_{k_1}) = f(\zeta_r) + f'(\zeta_r)(\delta_{\tilde{k}_1} + \beta_{21}\delta_{k_1}) + 0.5f''(\zeta_r)(\delta_{\tilde{k}_1} + \beta_{21}\delta_{k_1})^2$$

Since $\mathbf{rd}(\zeta_r) = \zeta_r$, so $f(\zeta_r) = \mathbf{Rd}(f(\zeta_r)) + \delta_\Sigma$.

Additionally, due to the quadraticity of f , it is true

$$f'(\zeta_r) = f'(\zeta) + f''(\zeta) \mathbf{rd}(\beta_{21}\tilde{k}_1(\zeta, \square)).$$

Thus,

$$f(\zeta + \beta_{21}k_1(\zeta, \square)) = \mathbf{Rd}(f(\zeta_r)) + \delta_\Sigma + \left(f'(\zeta) + f''(\zeta) \mathbf{rd}(\beta_{21}\tilde{k}_1(\zeta, \square))\right)(\delta_{\tilde{k}_1} + \beta_{21}\delta_{k_1}) + 0.5f''(\zeta_r)(\delta_{\tilde{k}_1} + \beta_{21}\delta_{k_1})^2.$$

So,

$$\begin{aligned} |f(\zeta + \beta_{21}k_1(\zeta, \square) - \mathbf{Rd}(f(\zeta_r)))| \\ \leq (|\delta_\Sigma| + m_1 + \beta_{21}m_2I_1(\square))(\delta_{\tilde{k}_1} + \beta_{21}\delta_{k_1}) + 0.5m_2(\delta_{\tilde{k}_1} + \beta_{21}\delta_{k_1})^2 \\ \leq |\delta_\Sigma| + (m_1 + \beta_{21}m_2I_1(\square))(\beta_{21}\Delta_1 + \Delta\sqrt{d}) + 0.5m_2(\beta_{21}\Delta_1 + \Delta\sqrt{d})^2 \end{aligned}$$

Multiplying both sides by $\square$ we obtain upper bound of numerical error for $k_2(\zeta, \square)$, in the following form:

$$|\delta_{k_2}| \leq \Delta_1 + \square(m_1 + \beta_{21}m_2I_1(\square))(\sqrt{d}\Delta + \beta_{21}\Delta_1) + 0.5\square m_2(\sqrt{d}\Delta + \beta_{21}\Delta_1)^2 = \Delta_2$$

where $\delta_{k_2} = k_2(\zeta, \square) - \mathbf{Rd}(k_2(\zeta, \square))$, $\mathbf{Rd}(k_2(\zeta, \square)) = \mathbf{rd}(\square \mathbf{Rd}(f(\zeta_r)))$.

As a result, for the calculation error with rounding by step, we obtain the estimate

$$|\bar{z}_{n+1} - \zeta_{n+1}| = |\bar{z}_{n+1} - \mathbf{Rd}(\zeta_n + p_1k_1(\zeta_n, \square) + p_2k_2(\zeta_n, \square))| \leq p_1\Delta_1 + p_2\Delta_2 + \sqrt{d}\Delta = \Delta_{rd}$$

Combining this estimate with the inequality

$$|z_{n+1} - \bar{z}_{n+1}| \leq (1 + Lh)|z_n - \zeta_n|$$

we get

$$\begin{aligned} |z_{n+1} - \bar{z}_{n+1} + \bar{z}_{n+1} - \zeta_{n+1}| &= |z_{n+1} - \zeta_{n+1}| \leq \Delta_{rd} + (1 + Lh)|z_n - \zeta_n| \leq \dots \\ &\leq \frac{(1 + Lh)^{n+1} - 1}{Lh} \Delta_{rd} < \frac{e^{LT} - 1}{Lh} \Delta_{rd}. \end{aligned}$$

Finally, $|z_n - \zeta_n| < \frac{e^{LT} - 1}{Lh} \Delta_{rd}$, for all $n = \overline{0, N}$.

Theorem is proved.

Similarly, for the Runge-Kutta scheme of the third order [6-11]

$$z_{n+1} = z_n + p_1k_1(z, h) + p_2k_2(z, \square) + p_3k_3(z, \square)$$

where

$$\begin{aligned} k_1(z, h) &= hf(z), k_2(z, h) = hf(z + \beta_{21}k_1(z, h)), \\ k_3(z, h) &= hf(z + \beta_{31}k_1(z, h) + \beta_{32}k_2(z, h)) \end{aligned}$$

and also:

$$p_1 + p_2 + p_3 = 1,$$

$$\begin{aligned} 2(\beta_{21}p_2 + (\beta_{31} + \beta_{32})p_3) &= 1, \\ 3(\beta_{21}^2p_2 + (\beta_{31} + \beta_{32})^2p_3) &= 1, \\ 6\beta_{21}\beta_{32}p_3 &= 1, \end{aligned}$$

following theorem is true.

Theorem. Let along with the assumption, conditions $z_n \in K$ and $\zeta_n \in K$ be satisfied for all $n = \overline{0, N}$. Then

$$|z_n - \zeta_n| < \frac{e^{Lh} - 1}{Lh} \Delta_{rd}, \quad n = \overline{0, N}$$

where

$$\begin{aligned} L &= p_1L_1 + p_2L_2 + p_3L_3; \quad L_1 = m_1; \\ L_2 &= (m_1 + \beta_{21}m_2I_1(h))(1 + \beta_{21}hL_1); \\ L_3 &= (m_1 + m_2(\beta_{31}I_1(h) + \beta_{32}I_2(\square)))(1 + \square(\beta_{31}L_1 + \beta_{32}L_2)), \\ I_1(\square) &= m_0h, \quad I_2(\square) = m_0h + \beta_{21}m_0m_1h^2 + \frac{\beta_{21}^2}{2}m_0^2m_2h^3. \end{aligned}$$

Δ_{rd} – calculation error with rounding at one step.

Proof. Similarly, let us introduce the operator

$$R[z] = z + p_1k_1(z, h) + p_2k_2(z, h) + p_3k_3(z, h)$$

Then the third order scheme will take the form $z_{n+1} = R[z_n]$.

Let's put $\bar{z}_{n+1} = R[\zeta_n]$

Then

$$\begin{aligned} R[x] - R[y] &= x - y + p_1(k_1(x, h) - k_1(y, h)) + p_2(k_2(x, h) - k_2(y, h)) \\ &\quad + p_3(k_3(x, h) - k_3(y, h)) \end{aligned}$$

Applying Adamar's lemma [4], we have

$$|k_1(x, h) - k_1(y, h)| \leq h \left\| \int_0^1 f'(\sigma_x^y) d\sigma \right\| |x - y| \leq m_1h|x - y|$$

where $\sigma_x^y = \sigma x + (1 - \sigma)y$.

Next, Adamar's lemma, as applied previously, is used to find for increments k_2 and k_3 . But this time, unlike the previous case, the argument of integrand can go beyond K . To get around this circumstance, using Taylor's formula results in:

$$\begin{aligned} |k_2(x, h) - k_2(y, h)| &\leq h \left\| \int_0^1 f'(\sigma_x^y + \beta_{21}\sigma k_1(x, h) + \beta_{21}(1 - \sigma)k_1(y, h)) d\sigma \right\| \\ &\quad \times |x - y + \beta_{21}(k_1(x, h) - k_1(y, h))|. \end{aligned}$$

Further,

$$\|f'(\sigma_x^y + \beta_{21}\sigma k_1(x, h) + \beta_{21}(1 - \sigma)k_1(y, h))\| \leq \|f'(\sigma_x^y)\| + \|f''(\sigma_x^y)(\beta_{21}\sigma k_1(x, h) + \beta_{21}(1 - \sigma)k_1(y, h))\| \leq m_1 + \beta_{21}m_2I_1(h).$$

Thus,

$$|k_2(x, h) - k_2(y, h)| \leq h(m_1 + \beta_{21}m_2I_1(h))(1 + \beta_{21}hL_1)|x - y| = L_2h|x - y|.$$

Further,

$$\begin{aligned}
 & |k_3(x, h) - k_3(y, h)| \\
 & \leq h \left\| \int_0^1 f'(\sigma_x^y + \beta_{31}\sigma k_1(x, h) + \beta_{32}\sigma k_2(x, h) + \beta_{31}(1 - \sigma)k_1(y, h) \right. \\
 & \quad \left. + \beta_{32}(1 - \sigma)k_2(y, h))d\sigma \right\| \\
 & \times |x - y + \beta_{31}(k_1(x, h) - k_1(y, h)) + \beta_{32}(k_2(x, h) - k_2(y, h))|.
 \end{aligned}$$

Let us separately evaluate the integrand

$$\begin{aligned}
 & \|f'(\sigma_x^y + \beta_{31}\sigma k_1(x, h) + \beta_{32}\sigma k_2(x, h) + \beta_{31}(1 - \sigma)k_1(y, h) + \beta_{32}(1 - \sigma)k_2(y, h))\| \leq \\
 & \|f'(\sigma_x^y)\| + \|f''(\sigma_x^y)\| \times |\beta_{31}\sigma k_1(x, h) + \beta_{32}\sigma k_2(x, h) + \beta_{31}(1 - \sigma)k_1(y, h) + \\
 & \beta_{32}(1 - \sigma)k_2(y, h)| \leq m_1 + m_2(\beta_{31}I_1(h) + \beta_{32}I_2(h)).
 \end{aligned}$$

That's why

$$|k_3(x, h) - k_3(y, h)| \leq h \left(m_1 + m_2(\beta_{31}I_1(h) + \beta_{32}I_2(h)) \right) (1 + h(\beta_{31}L_1 + \beta_{32}L_2)) |x - y| = L_3 h |x - y|$$

Thus

$$|R[x] - R[y]| \leq (1 + Lh) |x - y|,$$

where

$$L = p_1L_1 + p_2L_2 + p_3L_3.$$

Analogously, for the error in calculating the value of $k_1(\zeta, \square)$ one can obtain the estimate

$$|\delta_{k_1}| \leq \left(\square \sqrt{\sum_k \left(d + \sum_i \sum_{j \geq i} (b_{ij}^k + 1) \right)^2} + \sqrt{d} \right) \Delta = \Delta_1$$

where $\delta_{k_1} = \square f(\zeta) - \mathbf{rd}(\square \mathbf{Rd}(f(\zeta)))$.

And

$$|\delta_{k_2}| \leq \Delta_1 + \square(m_1 + \beta_{21}m_2I_1(\square))(\sqrt{d}\Delta + \beta_{21}\Delta_1) + 0.5\square m_2(\sqrt{d}\Delta + \beta_{21}\Delta_1)^2 = \Delta_2$$

where $\delta_{k_2} = k_2(\zeta, \square) - \mathbf{Rd}(k_2(\zeta, \square))$

$$\mathbf{Rd}(k_2(\zeta, \square)) = \mathbf{rd}(h \mathbf{Rd}(f(\zeta_r))).$$

Similarly,

$$\begin{aligned}
 & |k_3(\zeta, h) - \mathbf{Rd}(k_3(\zeta, h))| \\
 & \leq \Delta_1 + h \left(m_1 + m_2(\beta_{31}I_1(h) + \beta_{32}I_2(h)) \right) (\sqrt{d}\Delta + \beta_{31}\Delta_1 + \beta_{32}\Delta_2) \\
 & + 0.5hm_2(\sqrt{d}\Delta + \beta_{31}\Delta_1 + \beta_{32}\Delta_2)^2 = \Delta_3
 \end{aligned}$$

where $\mathbf{Rd}(k_3(\zeta, h))$ means successive roundings when calculating k_3 .

As a result, for the calculation error with rounding by step, we obtain the estimate

$$\begin{aligned}
 |\bar{z}_{n+1} - \zeta_{n+1}| & = |\bar{z}_{n+1} - \mathbf{Rd}(\zeta_n + p_1k_1(\zeta_n, h) + p_2k_2(\zeta_n, h) + p_3k_3(\zeta_n, h))| \\
 & \leq p_1\Delta_1 + p_2\Delta_2 + p_3\Delta_3 + \sqrt{d}\Delta = \Delta_{rd}
 \end{aligned}$$

Combining this estimate with the inequality

$$|z_{n+1} - \bar{z}_{n+1}| \leq (1 + Lh) |z_n - \zeta_n|$$

we get

$$\begin{aligned} |z_{n+1} - \bar{z}_{n+1} + \bar{z}_{n+1} - \zeta_{n+1}| &= |\bar{z}_{n+1} - \zeta_{n+1}| \leq \Delta_{rd} + (1 + Lh) |z_n - \zeta_n| \leq \dots \\ &\leq \frac{(1 + Lh)^{n+1} - 1}{Lh} \Delta_{rd} < \frac{e^{LT} - 1}{Lh} \Delta_{rd} \end{aligned}$$

Finally,

$$|z_n - \zeta_n| < \frac{e^{LT} - 1}{Lh} \Delta_{rd}$$

for all $n = \overline{0, N}$.

Theorem is proved.

4. CONCLUSION

The obtained error estimate for Runge-Kutta methods in solving the Cauchy problem for quadratic dynamical systems offers practical benefits. This finding highlights a common structure in the error behavior of these methods. By applying this estimate, users can more effectively control the accuracy

of numerical solutions, leading to more reliable simulations and improved understanding of the system's behavior. Further research could explore the application of this estimate to different types of dynamical systems and higher-order Runge-Kutta schemes.

References:

1. A. A. Azamov, "DN-Tracking Method to Prove the Existence of Limit Cycles. Differential Equations and Topology", Proceedings of the International Conference "The 100th Anniversary of Academician Lev Semenovich Pontryagin", Moscow, 87-88. (2008).
2. Uzbek Mathematical Journal, 2010, №4.
3. N. S. Bakhvalov, "Numerical Methods", Nauka, Moscow. 663 p. (1977).
4. И.Г. Петровский. Лекции по теории обыкновенных дифференциальных уравнений: Учебник для студ. мех.-мат. спец. ун-тов. // Под ред. А.Д. Мышкиса, О.А. Олейник. 7-ое изд., исправ. -М.: Изд-во МГУ, 1984.
5. Endre Suli and David F. Mayers. An introduction to numerical analysis.- Cambridge University Press, 2003.
6. Eugene Isaacson and Herbert Bishop Keller. Analysis of numerical methods.-Dover publications, New York, 1994.
7. Hildebrand, F.B, Introduction to numerical analysis, McGraw-Hill Book Co., New York, 1956.
8. J. C. Butcher. Numerical Methods for Ordinary Differential Equations. Second edition Wiley, 2008.
9. J. C. Butcher. Numerical Methods for Ordinary Differential Equations. Third edition Wiley, 2016.
10. L. F. Shampine, R. C. Allen and S. Pruess, *Fundamentals of Numerical Computing*, (1997) John Wiley & Sons.
11. R. England, *Error estimate for Runge-Kutta type solutions of ordinary differential equation*. Computer Journal 12 (1969) 166-170.

Uch quvuvchi va bir qochuvchili qochish differensial o'yini.

I.M.Yusupov

Z. M. Bobur nomidagi Andijon davlat universiteti, O'zbekiston, 170100, Andijan sh, Universitet ko'chasi 129 uy

Annotatsiya. Mazkur maqola uchta quvuvchi va bitta qochuvchining ishtirokidagi oddiy harakatli differensial o'yin tahliliga bag'ishlangan. O'yinchilarning boshqaruvlari Gronwall tipidagi energetik cheklovlar bilan chegaralangan. Agar qochuvchining holati hech bir vaqtda hech qaysi quvuvchining holatiga teng bo'lmasa, unda bunday vaziyat qochish mumkin bo'lgan holat sifatida qaraladi. Maqolada qochuvchi uchun shunday strategiya taklif etamizki, u quvuvchilarning dastlabki joylashuvlaridan qat'i nazar, ulardan muvaffaqiyatli qochishni ta'minlaydi.

Kalit so'zlar: Differensial o'yin, qochuvchi, quvuvchilar, boshqaruv, strategiya, monevr, Gronuoll chegara.

Аннотация: Данная статья посвящена анализу дифференциальной игры с простым движением с участием трёх преследователей и одного убегающего. Управление игроками ограничено энергетическими ограничениями типа Гронуолл. Если состояние убегающего никогда не равно состоянию ни одного преследователя, то такая ситуация рассматривается как состояние, в котором возможен уклонение. В статье мы предлагаем стратегию для убегающего, которая обеспечит успешное уклонение от преследователей независимо от их первоначального местоположения.

Ключевые слова: Дифференциальная игра, убегающий, преследователи, управление, стратегия, мonevr, граница Гронуолла.

Abstract: This article is devoted to the analysis of a differential game with simple motion involving three pursuers and one evader. The players' controls are subject to Gronwall-type constraints. If the state of the evader never coincides with the state of any pursuer, such a situation is considered as one in which evasion is possible. In this article, we propose a strategy for the evader that ensures successful evasion from the pursuers regardless of their initial positions.

Key words: Differential game, evader, pursuers, control, strategy, maneuver, Gronwall constraint.

I.Kirish.

Gronuoll turidagi cheklovlar differensial o'yinlarda muhim boshqaruvlardan biri sifatida qaraladi. Gronuoll boshqaruvli differensial o'yinlar [8, 9, 10, 11, 13]da keng o'rganilgan. Ayniqsa, [10, 11] ishlarda bitta quvuvchi va bitta qochuvchi ishtirokidagi differensial o'yinlar ya'ni quvish-qochish va "hayot chizig'i" turidagi modellar tahlil etilgan. Bu ishlarda o'yinchilarning boshqaruvlari Gronuoll turidagi boshqaruvlar bilan chegaralangan bo'lib, quvuvchi uchun II-strategiya deb ataluvchi parallel quvish strategiyasidan foydalanilgan. Shu bilan birga, qochuvchining optimal strategiyasi aniqlanib, tutish vaqtining optimal qiymati ham hisoblab topilgan. Bu natijalar Isaacs, Petrosyan, Pshenichnyi, Azamov kabi olimlarning fundamental tadqiqotlariga tayangan holda rivojlantirilgan.

Ko'p quvuvchili qochish differensial o'yinlari ham differensial o'yinlar nazariyada alohida o'rin tutadi. Bu borada [1–7] maqolalarda ko'p sonli O'zbekistonlik va xorijiy olimlarning ilmiy ishlari mavjud. Jumladan, [1, 4–7] maqolalarda boshqaruvlariga integral chegaralanishli ko'p quvuvchili differensial

o'yinlar tahlil qilingan. Shuningdek, [3]da bitta qochuvchi va m ta quvuvchidan iborat sodda harakatli, geometrik chegaralanishli differensial o'yin o'rganilgan. Bu ishda qochuvchining strategiyasi qurilib, uning muvaffaqiyatli qochib keta olishi asoslab berilgan.

Mazkur maqolada boshqaruvlariga Gronuoll chegaralanishli soda harakatli uchta quvuvchi va bir qochuvchili differensial o'yin ko'rib chiqiladi. Unda qochuvchi uchun samarali strategiya taklif etilib, bu strategiya yordamida u quvlovchilardan qochib ketishi mumkinligi isbotlanadi.

II. Masalaning qo'yilishi.

Ikki quvuvchi x_i , $i = 1, 2, 3$ va bitta qochuvchi y ning dinamikasi R^d da quyidagi differensial tenglamalar bilan tasvirlansin:

$$\begin{aligned} \dot{x}_i &= u_i, & x_i(0) &= x_{i0}, & i &= 1, 2, 3 \\ \dot{y} &= v, & y(0) &= y_0, \end{aligned} \quad (1)$$

bu yerda $x_i, y, x_{i0}, y_0 \in \mathbb{R}^d$, $d \geq 2$, u_i va v mos ravishda i -chi quvuvchi $i = 1, 2, 3$ va qochuvchi y ning boshqaruv parametrlari.

Ta'rif 1. Borel o'lchovli $u_i(t)$ va $v(t)$, $t \geq 0$ funksiyalar quyidagi tengsizliklarni qanoatlantirsa:

$$|u_i(t)|^2 \leq \rho_i^2 + 2b \int_0^t |u_i(s)|^2 ds, \quad t \geq 0, \quad (2)$$

$$|v(t)|^2 \leq \sigma^2 + 2b \int_0^t |v(s)|^2 ds, \quad t \geq 0, \quad (3)$$

mos ravishda quvuvchilar x_i , $i = 1, 2, 3$ va qochuvchi y ning joiz boshqaruvlari deyiladi, bu yerda ρ_i , σ , $\rho_i \leq 1$, $\sigma > 1$ va b berilgan musbat sonlardir.

Biz x_i -chi quvuvchi va y qochuvchining barcha ruxsat etilgan boshqaruvlari to'plamlarini mos ravishda U_i va V bilan belgilaymiz.

(1) tenglamalarini yechib, biz quvuvchilar va qochuvchining joiz boshqaruvlari $u_i(\cdot) \in U_i$ va $v(\cdot) \in V$ ga mos keladigan traektoriyalarini quyidagi tenglamalar bilan aniqlaymiz:

$$x_i(t) = x_{i0} + \int_0^t u_i(s) ds, \quad i = 1, 2, 3 \quad y(t) = y_0 + \int_0^t v(s) ds,$$

mos ravishda.

Lemma 1. Agar [12]

$$|\omega(t)|^2 \leq \rho^2 + 2b \int_0^t |\omega(s)|^2 ds$$

bo'lsa, u holda $|\omega(t)| \leq \rho e^{bt}$ bo'ladi, bu yerda $\omega(t)$, $t \geq 0$, o'lchovli funksiya va ρ berilgan musbat sonidir.

Lemma 1 ga ko'ra, o'yinchilarning joiz boshqaruvlari $u_i(\cdot) \in U$ va $v(\cdot) \in V$ uchun bizda quyidagilar mavjud:

$$|u_i(t)| \leq \rho_i e^{bt}, \quad i = 1, 2, 3 \quad |v(t)| \leq \sigma e^{bt} \quad (4)$$

Biz qochuvchi strategiyasi uchun ta'rif beramiz.

Ta'rif 2. Uzluksiz $V : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{3d} \rightarrow \mathbb{R}^d$

$$(t, \theta_1, \theta_2, y, x_1, x_2, u_1, u_2) \rightarrow V(t, \theta_1, \theta_2, y, x_1, x_2, u_1, u_2)$$

funksiya, qochuvchining strategiyasi deb ataladi, agar quvuvchilarning ixtiyoriy joiz $u_1 = u_1(t)$, $u_2 = u_2(t)$ boshqaruvlari uchun

$$v = V(t, \theta_1, \theta_2, y, x_1, x_2, u_1, u_2)$$

da, (1) boshlang'ich qiymat masalasi yagona $(y(t), x_1(t), x_2(t))$ $\theta_1 = \theta_1(t)$, $\theta_2 = \theta_2(t)$ yechimiga ega bo'lsa va quyidagi shart bajarilsa:

$$|V(t, \theta_1, \theta_2, y, x_1, x_2, u_1, u_2)|^2 \leq \sigma^2 + 2b \int_0^t |V(s, \theta_1, \theta_2, y, x_1, x_2, u_1, u_2)|^2 ds, \quad t \geq 0.$$

bu yerda $\theta_1(t)$ va $\theta_2(t)$, $t \geq 0$ berilgan funksiyalardir.

Quvuvchilarning harakatlari ixtiyoriy, ya'ni quvuvchilar qochuvchi strategiyani qo'llagan vaqtda har qanday joiz boshqaruvlarni qo'llaydilar.

Ta'rif 3. Biz (1) o'yinda qochish mumkin deymiz, agar qochuvchining shunday V strategiyasi mavjud bo'lsaki, quvuvchilarning har qanday boshqaruvlari uchun barcha $t \geq 0$ va $i = 1, 2, 3$ uchun $x_i(t) \neq y(t)$ tengsizlik o'rinli bo'lsa.

Muammo: (1) o'yinida qochish mumkinligini ta'minlash uchun qochuvchining V strategiyasini qurish.

Qochish o'yini (1) ni $\mathbb{R}^2$ da, ya'ni $d = 2$ bo'lganda ko'rib chiqish kifoya.

III. Asosiy natija.

Bir quvuchi va bir qochuvchili hol. Biz (1) differensial o'yin uchun quyidagi parametrlar va tushunchalarni kiritib olaylik.

α va a sonlarini quyidagi shartlarni qanoatlantiradigan qilib tanlaymiz:

$$0 < \alpha < \frac{1}{2}(\sigma - 1), \quad 0 < a < |x_0 - y_0|. \quad (5)$$

Qochuvchi strategiya qoʻllaydi, quvuvchilar esa ixtiyoriy boshqaruv $u_i(t) = (u_{i1}(t), u_{i2}(t))$, $t \geq 0$ dan foydalanadi. Quvuvchining trayektoriyasi esa $x_i(t) = (x_{i1}(t), x_{i2}(t))$ ga teng.

Keyingi bosqichda, qochuvchining qochishni taʼminlaydigan strategiyasini quramiz. Avvalo, qochuvchi Oy -oʻqi boʻylab quyidagicha harakat qiladi:

$$V_0(t) = (0, \sigma e^{bt}), \quad t \in [0, \tau), \quad (6)$$

bu yerda $t = \tau : |x_i(t) - y(t)| = a$ boʻlgan birinchi vaqt boʻlib, u quvuvchilardan qochuvchiga a masofagacha yaqinlashish vaqti deb yuritiladi.

Bu yerda ikki holat mavjud. Birinchisi - τ vaqt mavjud boʻlmasligi mumkin. Agar shunday boʻlsa, u holda $|x_i(t) - y(t)| > a$ har qanday $t \geq 0$ uchun bajariladi va bu holatda $x_i(t) \neq y(t)$ boʻlib qoladi, yaʼni qochish har doim ($t \geq 0$ uchun) mumkin boʻladi. Endi vaqt τ mavjud deb faraz qilamiz. Biz τ' vaqtini quyidagicha aniqlangan vaqt deb olamiz:

$$\tau' = \frac{1}{b} \ln \left(e^{b\tau} + \frac{4ab}{\sigma - 1} \right).$$
 Biz (1) differensial oʻyinda

quvuvchilarning biri qochuvchi bilan a masofaga yaqinlashsa ikkinchi quvlovchi τ' vaqtdan keyin a masofaga yaqinlashgan holni koʻramiz.

Biz (1) differensial oʻyinni quyidagi bitta quvuvchi va bitta qochuvchi boʻlgan xolini koʻrib chiqaylik :

$$\begin{aligned} \dot{x} &= u, & x(0) &= x_0, \\ \dot{y} &= v, & y(0) &= y_0, \end{aligned} \quad (7)$$

bu yerda $x_0 \neq y_0$, va $u = (u_1, u_2)$, $v = (v_1, v_2)$ - quvuvchi va qochuvchining boshqaruv parametrlari.

Biz qochuvchi $V(t) = (V_1(t), V_2(t))$ uchun $[\tau, \tau')$ oraligʻida quyidagi strategiyani taklif qilamiz:

$$V_1(t) = \begin{cases} |u_1(t)| + \alpha e^{bt}, & x_1(\tau) \leq y_1(\tau) \\ -(|u_1(t)| + \alpha e^{bt}), & x_1(\tau) > y_1(\tau) \end{cases}, \quad V_2(t) = \sqrt{\sigma^2 e^{2bt} - V_1^2(t)}. \quad (8)$$

Aniqki,

$$|V_1(t)| = |u_1(t)| + \alpha e^{bt} \leq e^{bt} + \frac{1}{2}(\sigma - 1)e^{bt} \leq \sigma e^{bt},$$

shuningdek $V_2(t)$ ham aniqlangan

$[\tau, \tau')$ vaqt oraligʻida aniqlangan.

$V(t) = (V_1(t), V_2(t))$ strategiyasi qochuvchining quvuvchiga nisbatan manevr harakati deb ataladi. τ' vaqtdan boshlab, qochuvchi quyidagi tezlik bilan harakatda davom etadi:

$$V_0(t) = (0, \sigma e^{bt}), \quad t \geq \tau'. \quad (9)$$

Lemma 2. Agar qochuvchi (6), (8) va (9) strategiyalaridan foydalansa, u holda:

$$|y(t) - x(t)| \geq a, \quad 0 \leq t \leq \tau, \quad (10)$$

$$|y(t) - x(t)| > \frac{\alpha a}{2\sigma}, \quad \tau \leq t \leq \tau', \quad (11)$$

$$y_2(t) - x_2(t) \geq a, \quad t \geq \tau'. \quad (12)$$

tengsizliklar oʻrinli.

Lemma isbotini Yusupov I.M va Zokirjonov S.M larning [13] maqolasida koʻrishingiz mumkin.

Koʻp quvuvchi va bir qochuvchili xol.

Teorema. (1) oʻyinda oʻyinchilarning istalgan boshlangʻich holatlari uchun qochish mumkin.

Teoremani isbotlash uchun qochishni taʼminlovchi qochuvchining strategiyasini tuzamiz. Isbotni bir nechta kichik boʻlimlarga ajratamiz.

Parametrlarning taʼriflari. Quyidagi ifodalarni qanoatlantiruvchi musbat sonlarni tanlaymiz.

$$0 < \alpha < \min \left\{ 1, \frac{1}{2}(\sigma - 1) \right\},$$

$$0 < a_1 < \min_{i=1, \dots, m} |y_0 - x_{i0}|, \quad \beta = \frac{(\sigma - 1)\alpha}{128\sigma^2 e^{bT}},$$

bu yerda $T = \tau_1 + \frac{8a_1}{\sigma - 1}$. $\beta < \frac{1}{2}$ chunki

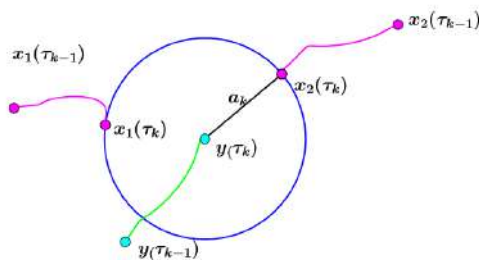
$$e^{bT} > 1, \alpha < \frac{1}{2}(\sigma - 1).$$

a_1 sonini tanlangach, biz $a_{k+1} = \beta a_k$, $k = 1, 2, \dots$, tenglik orqali kamayuvchi $\{a_k\}_{k=1}^{\infty}$

musbat sonlar ketma-ketligini aniqlab olamiz. Endi yaqinalshish vaqtlarini aniqlashtirib olamiz. Tasavvur qilaylik qochuvchi biror strategiya ostida harakatlanayotgan bo'lsin. $t = \tau_1 > 0$ soni birinchi marta $|x_{s_1}(\tau_1) - y(\tau_1)| = a_1, s_1 \in \{1, 2, 3\}$ tenglik va barcha $0 \leq t < \tau_1, i = 1, 2, 3$ uchun $|x_i(t) - y(t)| > a_1$ tengsizlik bajariladigan vaqt bo'lsin. Ushbu τ_1 vaqt x_{s_1} quvuvchining y qochuvchiga a_1 -yaqinlashish vaqti deb ataladi.

Umumiy qilib aytganda, agar $\tau_1, \tau_2, \dots, \tau_{k-1}, \tau_1 < \tau_2 < \dots < \tau_{k-1}$ sonlari mos ravishda $a_1, a_2, \dots, a_{k-1}$ -yaqinlashish vaqtlari bo'lsa, u holda biz $|x_{s_k}(\tau_k) - y(\tau_k)| = a_k, s_k \in \{1, 2, 3\}$ tenglik va barcha $0 \leq t < \tau_k, i = 1, 2, 3$ uchun $|x_i(t) - y(t)| > a_k$ tengsizlik bajariladigan qilib $t = \tau_k > \tau_{k-1}$ a_k - yaqinlashish vaqtini aniqlab olamiz.

Shuni ta'kidlab o'tish kerakki, τ_k vaqt bir vaqtning o'zida bir nechta quvuvchilarda sodir bo'lishi mumkin. Misol uchun, 1-Rasm da a_k -yaqinlashish vaqti birdaniga x_1, x_2 quvuvchilarda sodir bo'lgan. U holda umumiylikka ziyon keltirmagan holda biz bu quvuvchilarning ixtiyoriy birini tanlab uni x_k quvuvchi deb olamiz.



1-Rasm. Ikki quvuvchi a_k -yaqinlashish vaqti.

x_k quvuvchi uchun τ_k vaqtdan tashqari keyinchalik $\tau_{k'}, \tau_{k''}, \dots$, yaqinlashish vaqtlari ham sodir bo'lishligi muhim hisoblanadi. Aytaylik

$$\tau'_k = \frac{1}{b} \ln \left(e^{b\tau_k} + \frac{4a_k b}{\sigma - 1} \right), k = 1, 2, \dots, \tau_0 = 0, \tau'_0 = \infty.$$

E'tibor bering, $\tau'_1, \tau'_2, \tau'_3, \dots$ ketma-ketlik o'suvchi bo'lishi shart emas.

Tasdiq. Har bir $k = 1, 2, \dots$ uchun quyidagi tengsizlik o'rinli

$$\tau'_k - \tau_k \leq \frac{4a_k}{\sigma - 1}.$$

Isbot. τ'_k aniqlanishi, $e^{b\tau} > 1$ va $\ln(1+x) < x$ $x > 0$ tengsizliklarga ko'ra quyidagiga ega bo'lamiz:

$$\begin{aligned} \tau'_k - \tau_k &= \frac{1}{b} \ln \left(e^{b\tau_k} + \frac{4a_k b}{\sigma - 1} \right) - \tau_k = \frac{1}{b} \ln \left(1 + \frac{4a_k b}{e^{b\tau_k} (\sigma - 1)} \right) \\ &\leq \frac{1}{b} \ln \left(1 + \frac{4a_k b}{\sigma - 1} \right) \leq \frac{4a_k}{\sigma - 1}. \end{aligned}$$

Qochuvchi uchun strategiya. Umumiylikka ziyon keltirmagan holda $y_0 = (0, 0)$ bo'lsin. Aytaylik $u_i(t), i = 1, 2, 3$, quvuvchilarning ixtiyoriy joiz boshqaruvlari bo'lsin. Qochuvchi uchun biz quyidagi strategiyani taklif qilamiz.

Dastlab qochuvchi Oyo'q bo'ylab $\tau_0 = 0$ vaqtdan boshlab $v(t) = V_0(t) = (0, \sigma e^{bt})$ tezlik bilan harakatlanadi. Agar a_1 -yaqinlashish vaqti sodir bo'lmasa, ya'ni barcha $i = 1, 2, 3$ va $t \geq 0$ uchun $|y(t) - x_i(t)| > a_1$ bo'lsa, u holda, $x_i(t) \neq y(t), t \geq 0, i = 1, 2, 3$ ekanligi ko'rinib turibdi, bu esa o'yinda qochish imkoniyatini ko'rsatadi. Agar a_1 -yaqinlashish vaqti sodir bo'lsa, u holda quyidagi $k = 1, 2, \dots$ uchun

$V_k(t) = (V_{k1}(t), V_{k2}(t))$ manyovrlarni aniqlaymiz

$$V_{k1}(t) = \begin{cases} |u_{k1}(t)| + \alpha e^{bt}, & x_{k1}(\tau_k) \leq y_1(\tau_k), \\ -(|u_{k1}(t)| + \alpha e^{bt}), & x_{k1}(\tau_k) > y_1(\tau_k), \end{cases} \quad V_{k2}(t) = \sqrt{\sigma^2 e^{2bt} - V_{k1}^2(t)}. \quad (14)$$

a_k -yaqinlashish vaqti sodir bo'lsin. U holda qochuvchining strategiyasi quyidagicha bo'ladi:

(i) agar $[\tau_k, \tau'_k)$ vaqt oralig'ida τ_{k+1} sodir bo'lsa, u holda $[\tau_k, \tau_{k+1})$ da $v(t) = V_k(t)$ bo'ladi,

(ii) agar $[\tau_k, \tau'_k)$ vaqt da τ_{k+1} sodir bo'lmasa, $[\tau_k, \tau'_k)$ da $v(t) = V_k(t)$ bo'ladi,

(iii) agar $[\tau'_k, \infty)$ vaqt da τ_{k+1} sodir bo'lsa, u holda $[\tau'_k, \tau_{k+1})$ da $v(t) = V_0(t)$

(iv) agar τ_{k+1} vaqt hech qachon sodir bo'lmasa, u holda $[\tau'_k, \infty)$ da $v(t) = V_0(t)$ bo'ladi.

Endi $\theta_1(t)$ va $\theta_2(t)$ funksiyalarini aniqlaymiz. Biror $K \geq 0$ uchun $\theta_1(t) = \max_{\tau_k \leq t} \tau_k = \tau_K$ bo'lsin. Demak t dan oshib ketmaydigan τ_k larning eng kattasi τ_K . So'ngra $\theta_2(t) = \tau'_K$ ni aniqlaymiz. Masalan, agar $\tau_0 \leq t < \tau_1$ bo'lsa, $K = 0$ va agar $\tau_1 \leq t < \tau_2$ bo'lsa, $K = 1$.

(i) - (iv) strategiya tavsifiga ko'ra, biz ikkita mumkin bo'lgan holatni o'rganamiz.

A hol. Agar $\tau_1 < \tau_2 < \dots < \tau_{k_1}$ ($\tau_2 < \tau'_1, \tau_3 < \tau'_2, \dots, \tau_{k_1} < \tau'_{k_1-1}$) k_1 ta yaqinlashish vaqtlari sodir bo'lsin-u, lekin τ_{k_1+1} yaqinlashish vaqti $[\tau_{k_1}, \tau'_{k_1})$ vaqt oralig'ida sodir bo'lmasin. U holda qochuvchi $[\tau_1, \tau'_{k_1})$ vaqt oralig'ida $x_1, x_2, \dots, x_{k_1}$ quvuvchilarning guruh hujumi ostida deyiladi va bu guruh hujum τ'_{k_1} vaqtda yakunlanadi.

(i) - (iv) holatlarga ko'ra, $[\tau_0, \tau_{k_1}]$ oralig'ida qochuvchining strategiyasi quyidagicha yozilishi mumkin:

$$v(t) = \begin{cases} V_0(t), & \tau_0 \leq t < \tau_1, \\ V_k(t), & \tau_k \leq t < \tau_{k+1}, \quad k = 1, 2, \dots, k_1 - 1, \\ V_{k_1}(t), & \tau_{k_1} \leq t < \tau'_{k_1}, \end{cases} \quad (15).$$

Birinchi guruh hujumi tugagandan so'ng qochuvchi ta'qibchilarning navbatdagi guruh hujumi boshlanguncha $v(t) = V_0(t)$ tezlik bilan harakatlanadi.

A holatda yaqinlashish vaqtlarini ta'riflaridan $\tau_k < \tau_{k+1}$ va $\tau_{k+1} < \tau'_k$, $k = 1, \dots, k_1 - 1$ bo'di va bundan quyidagi ifodaga ega bolamiz:

$$[\tau_1, \tau_{k_1}) = \bigcup_{k=1}^{k_1-1} [\tau_k, \tau_{k+1}) \cup [\tau_{k_1}, \tau'_{k_1}) \subset \bigcup_{k=1}^{k_1} [\tau_k, \tau'_k). \quad (16)$$

O'z-o'zidan savollar tug'iladi: bitta guruh hujumida bitta ta'qibchining o'zi bir necha marta yaqinlashishi mumkinmi? Guruhli hujumda yaqinlashishlar soni chekli bo'ladimi? Biz shuni aniqlaymizki, qochuvchilar strategiyasining (i) - (iv) tavsifi quyida B holatda ko'rib chiqiladigan bitta guruh hujumida cheksiz ko'p marta yaqinlashish vaqtlarining mavjudligi inkor qiladi.

B holat. Aytaylik x_1, x_2, x_3 quvuvchilarning y qochuvchiga nisbatan cheksizta $\tau_1, \tau_2, \dots$ muvaffaqiyatli yaqinlashish vaqtlari sodir bo'lsin. Bunda yaqinlashish vaqtlari $\tau_k < \tau_{k+1} < \tau'_k$, $k =$

$1, 2, \dots$ tengsizlikni qanoatlantiradi. U holda $[\tau_k, \tau_{k+1}) \subset [\tau_k, \tau'_k)$ bo'lganligi uchun barcha $n \geq 1$ uchun biz quyidagiga egamiz

$$[\tau_1, \tau_n) = \bigcup_{k=1}^{n-1} [\tau_k, \tau_{k+1}) \subset \bigcup_{k=1}^{n-1} [\tau_k, \tau'_k). \quad (17)$$

Tasdiqqa ko'ra,

$$\begin{aligned} \tau_n - \tau_1 &< \sum_{k=1}^{n-1} (\tau'_k - \tau_k) \leq \sum_{k=1}^{\infty} \frac{4a_k}{\sigma - 1} \\ &= \frac{4a_1}{(\sigma - 1)(1 - \beta)} < \infty. \end{aligned}$$

Bu esa o'suvchi τ_n ketma-ketlikning chegaralanganligini bildiradi. Natijada, $\tau_{\infty} = \lim_{n \rightarrow \infty} \tau_n$ mavjud. Shuni ta'kidlash kerakki, B holatda (17) tenglik ixtiyoriy $n \geq 1$ uchun o'rinli va (17) ifodada $n \rightarrow \infty$ limitga o'tsak quyidagi ifodaga ega bo'lamiz:

$$[\tau_1, \tau_{\infty}) \subset \bigcup_{k=1}^{\infty} [\tau_k, \tau'_k). \quad (18)$$

Qochuvchining strategiyasini tavsiflashning (i) bandiga ko'ra, uni $[\tau_1, \tau_{\infty})$ intervalda quyidagicha yozish mumkin:

$$v(t) = V_k(t), \quad t \in [\tau_k, \tau_{k+1}), \quad \tau_k <$$

Bundan keyin biz A holatdagi τ_{k_1}' va B holatdagi τ_{∞} ifodalash uchun $\bar{\tau}$ dan foydalanamiz. Boshqacha aytganda, birinchi guruh hujumi A va B holatlarda ham $\bar{\tau}$ da tugaydi. Biz (16), (18) va

$$\beta \leq \frac{1}{2} \text{ dan quyidagiga ega bo'lamiz } \bar{\tau} - \tau_1 \leq \sum_{k=1}^{\infty} (\tau'_k - \tau_k) \leq \sum_{k=1}^{\infty} \frac{4a_k}{\sigma - 1}.$$

Shuning uchun,

$$\bar{\tau} \leq \tau_1 + \frac{8a_1}{\sigma - 1} = T. \quad (20)$$

Birinchi guruh hujumida ixtiyoriy a_k - yaqinlashish vaqti uchun $\tau_k < \bar{\tau}$ ga ega bo'lamiz, shuning uchun (20) ga ko'ra $\tau_k < T$.

Biz τ_1 dan $\bar{\tau}$ bo'lgan vaqtda birinchi guruh hujumining o'ziga xos xususiyatlarini ko'rib chiqamiz. Shuni ta'kidlash joizki, $\bar{\tau}$ keyin yana bir guruh hujumi sodir bo'lishi mumkin va ikkinchi guruh hujumini ham xuddi shunday tahlil qilish mumkin.

Ohirgi bo'limdagi xulosalar shuni ko'rsatadiki, B holat yuzaga kelishi mumkin emas.

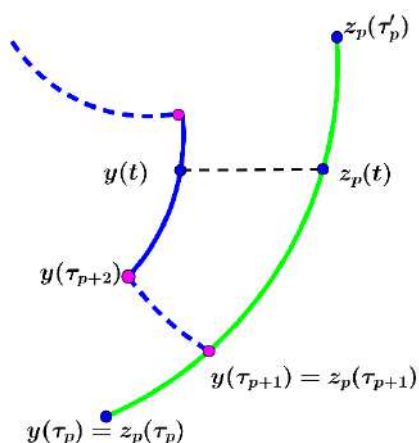
Bundan tashqari, keyingi kichik bo'limlarda qandaydir ta'qibchi bir necha guruhli hujumlarda qatnasha oladimi, guruhli hujumlar soni chekli bo'ladimi kabi savollarga javob beramiz.

Qochuvchi va soxta qochuvchi o'rtasidagi masofani baholash

A holatda har qanday $p \in \{1, 2, \dots, k_1\}$ va B holatda har qanday p musbat butun son uchun x_p quvlovchining y qochuvchiga a_p -yaqinlashish vaqtini ko'rib chiqamiz. Biz $t \geq \tau_p$ vaqt uchun $x_p(t)$ va $y(t)$ orasidagi masofani baholamoqchimiz. Ushbu baholashga erishish uchun biz harakatini $t \in [\tau_p, \tau'_p]$ oraliqda quyidagi tenglama bilan tavsiflaydigan soxta qochuvchini (SQ) kiritamiz:

$$\dot{z}_p = w_p, z_p(\tau_p) = y(\tau_p),$$

bu yerda - SQ ning boshqaruv parametri va



2-rasm: Qochuvchi va SQ ning trayektoriyalari.

U holda (11) ga ko'ra quyidagiga ega bo'lamiz

$$|z_p(t) - x_p(t)| > \frac{\alpha a_p}{2\sigma} \quad \text{for } \tau_p \leq t \leq \tau'_p. \quad (22).$$

Bundan tashqari (12) ga ko'ra quyidagi ham o'rinli

Shuni ta'kidlash kerakki, SQ faqat vaqt oraliqida harakatlanadi. Uning boshlang'ich holati

haqiqiy qochuvchining boshlang'ich holati bilan bir xil.

A holatda (15) ga ko'ra qochuvchi, $[\tau_p, \tau_{k_1}]$ $p \leq k_1$, oraliqida quyidagi strategiyadan foydalanadi

bu yerda

$$\tau_k < \tau_{k-1}', \quad k = p+1, p+2, \dots, k_1; \quad (25).$$

va B holatda (19) ko'ra qochuvchi oraliqda quyidagi strategiyadan foydalanadi $v(t) = V_k(t), t \in [\tau_k, \tau_{k+1}), \tau_k <$

Biz $[\tau_p, \bar{\tau}]$ da $z_p(t)$ soxta qochuvchi va qochuvchi orasidagi masofani baxolash uchun

$$[\tau'_p, t) \subset [\tau_{p+1}, t) \subset \bigcup_{k \geq p+1, \tau_k \leq t} [\tau_k, \tau'_k), \quad \tau_{p+1} \leq t \leq \bar{\tau}, \quad (27)$$

ifodadan foydalanib quyidagi lemmani keltiramiz. Bu yerda A holda va B holda.

3-lemma. Agar qochuvchi A holatda (31) (21). va B holatda (33) strategiyalardan foydalansa, u holda

$$|y(t) - z_p(t)| \leq \frac{32\sigma a_{p+1} e^{bT}}{\sigma - 1} \quad \text{for } t \in [\tau_p, \tau'_p] \cap [\tau_p, \bar{\tau}]. \quad (28).$$

Lemmaning isbotida ikkita holat borligini ta'kidlab o'tamiz: 1) $\tau'_p \leq \bar{\tau}$ ya'ni $t \in [\tau_p, \tau'_p]$, va 2) $\tau'_p > \bar{\tau}$ ya'ni $t \in [\tau_p, \bar{\tau}]$.

Isbot. Agar oraliqda $[\tau_p, \tau'_p]$ yaqinlashish vaqti bo'lmasa, u holda (21) va (24) $v(t) = w_p(t) = V_p(t), t \in [\tau_p, \tau'_p]$ dan foydalanib, $y(\tau_p) = z(\tau_p)$ hosil qilamiz, bundan esa $y(t) = z_p(t), \tau_p \leq t \leq \tau'_p$ hosil qilamiz. Bu hol uchun (28) to'g'ri ekan.

Endi $[\tau_p, \tau'_p]$ oraliqda bir yoki bir necha marta yaqinlashish $\tau_{p+1}, \tau_{p+2}, \dots$ sodir bo'lsin. 6-rasmda qochuvchining trayektoriyasi yoylari aniq manyovrlarga mos keladi. Biz (21) va (24) larga ko'ra $v(t) = w_p(t) = V_p(t)$ bo'lganligi uchun $t \in [\tau_p, \tau_{p+1})$ oraliqda $y(t) = z_p(t)$ ekanligidan (28) tengsizlik o'rinli bo'ladi.

$$z_{p+2}(\tau'_{p+2}) - x_{p+2}(\tau'_{p+2}) > a_p. \quad (23)$$

Biz $t \in [\tau_{p+1}, \tau'_{p+1})$ uchun quyidagiga ega bo'lamiz:

$$|y(t) - z_p(t)| = \left| y(\tau_p) + \int_{\tau_p}^{\tau_{p+1}} V_p(s) ds + \int_{\tau_{p+1}}^t v(s) ds - z(\tau_p) - \int_{\tau_p}^{\tau_{p+1}} V_p(s) ds - \int_{\tau_{p+1}}^t V_p(s) ds \right| = \frac{\alpha a_p}{2\sigma} - \frac{32\sigma e^{bT} \beta a_p}{\sigma - 1} = \frac{\alpha a_p}{4\sigma},$$

$$= \left| \int_{\tau_{p+1}}^t (v(s) - V_p(s)) ds \right| \leq \int_{\tau_{p+1}}^t |v(s) - V_p(s)| ds \leq \int_{\tau_{p+1}}^t 2\sigma e^{-\beta s} ds [\tau_p, \tau'_p] \cap [\tau_p, \bar{\tau}].$$

U holda

$$= \frac{2\sigma}{b} (e^{bt} - e^{b\tau_{p+1}}) \leq \frac{2\sigma e^{b\tau_{p+1}}}{b} (e^{b(t-\tau_{p+1})} - 1). \quad (29)$$

$$|y(t) - x_p(t)| > \frac{\alpha a_p}{4\sigma} \geq \beta a_p = a_{p+1},$$

Bu yerda

$$|v(s) - V_p(s)| \leq |v(s)| + |V_p(s)| \leq 2\sigma e^{bs}$$

tengsizlikdan foydalanildi. Shartga ko'ra $\tau_{p+1} \leq t \leq \tau'_p$, $t \leq \bar{\tau}$, (27) o'rinli va shuning

uchun biz Tasdiq va $\beta \leq \frac{1}{2}$ ko'ra quyidagiga

egamiz

$$t - \tau_{p+1} \leq \sum_{k \geq p+1, \tau_k \leq t} (\tau'_k - \tau_k) \leq \sum_{k \geq p+1} \frac{4a_k}{\sigma - 1} = \frac{4a_{p+1}}{(\sigma - 1)(1 - \beta)} \leq \frac{8a_{p+1}}{\sigma - 1}. \quad (30)$$

Natijada (36), (37) va $\tau_k \leq \bar{\tau} \leq T$ lardan foydalsak quyidagini hosil qilamiz:

$$|y(t) - z_p(t)| \leq \frac{2\sigma e^{bT}}{b} \left(e^{\frac{8a_{p+1}b}{\sigma - 1}} - 1 \right).$$

U holda quyidagi $e^x - 1 < 2x$, $0 < x < 1$,

$$x = \frac{8a_{p+1}}{\sigma - 1} \text{ tengsizliklardan foydalanib}$$

$$|y(t) - z_p(t)| \leq \frac{2\sigma e^{bT}}{b} \left(e^{\frac{8a_{p+1}b}{\sigma - 1}} - 1 \right) < \frac{2\sigma e^{bT}}{b} \frac{16a_{p+1}b}{\sigma - 1} = \frac{32\sigma a_{p+1} e^{bT}}{\sigma - 1} t \geq \tau'_p \text{ uchun}$$

hosil qilamiz. Lemma isbotlandi.

Quvuvchi va qochuvchi orasidagi masofani

baholash. Endi x_p quvuvchi va y qochuvchi orasidagi masofani baholaymiz.

Lemma 4. Qochuvchi A holda (24) va B holda (26) strategiyani qo'llasin. U holda

$$|y(t) - x_p(t)| > a_{p+1}, \quad \tau_p \leq t \leq \bar{\tau} \quad (31)$$

va agar $\tau'_p \leq \bar{\tau}$ bo'lsa, u holda

$$y_2(t) - x_{p2}(t) > a_{p+1}, \quad t \geq \bar{\tau}. \quad (32)$$

Isbot. (31) ni isbotlash uchun biz (22) va (28) tengsizliklarning $[\tau_p, \tau'_p] \cap [\tau_p, \bar{\tau}]$ da to'g'ri ekanligini hisobga olib quyidagiga ega bo'lamiz

$$|y(t) - x_p(t)| \geq |x_p(t) - z_p(t)| - |z_p(t) - y(t)| > \frac{\alpha a_p}{2\sigma} - \frac{32\sigma e^{bT} a_{p+1}}{\sigma - 1}$$

$$t \in [\tau_p, \tau'_p] \cap [\tau_p, \bar{\tau}]. \quad (33)$$

Bu bilan agar $\tau'_p \geq \bar{\tau}$ bo'lsa, (31) o'rinli ekan. $\tau'_p < \bar{\tau}$ bo'lgan holda (43) tengsizlik (31) tengsizlikning $[\tau_p, \tau'_p]$ kesmada o'rinli ekanligini bildiradi. Demak, agar $\tau'_p < \bar{\tau}$ bo'lsa, $\tau'_p \leq t \leq \bar{\tau}$ uchun (31) tengsizlikni isbotlash kifoya.

(23) ga ko'ra $z_{p2}(\tau'_p) - x_{p2}(\tau'_p) > a_p$ tengsizlik va (28) ga ko'ra $|y(\tau'_p) - z_p(\tau'_p)| \leq \frac{32\sigma e^{bT}}{\sigma - 1} a_{p+1}$ tengsizliklarning o'rinli ekanligi hamda α va β larning ta'riflariga ko'ra quyidagiga egamiz:

$$y_2(\tau'_p) - x_{p2} = (z_{p2}(\tau'_p) - x_{p2}(\tau'_p)) + (y_2(\tau'_p) - z_{p2}(\tau'_p))$$

$$\geq a_p - |y(\tau'_p) - z_p(\tau'_p)| > a_p - \frac{32\sigma e^{bT}}{\sigma - 1} a_p \beta > \frac{1}{2} a_p. \quad (34)$$

$t \geq \tau'_p$ uchun (34) tengsizlik va $u_{p2}(s) \leq e^{bs}$ tengsizlikdan foydalsak, quyidagi tengsizlik hosil bo'ladi:

$$y_2(t) - x_{p2}(t) = y_2(\tau'_p) - x_{p2}(\tau'_p) + \int_{\tau'_p}^t v_2(s) ds - \int_{\tau'_p}^t u_{p2}(s) ds > \frac{1}{2} a_p + \int_{\tau'_p}^t v_2(s) ds - \frac{1}{b} (e^{bt} - e^{b\tau'_p}). \quad (35)$$

A. Barcha $t \geq 0$ uchun $v_2(t) > 0$ bo'lganligi sababli, $\int_{\tau'_p}^t v_2(s) ds \geq 0$. Shuningdek, (27) va yuqorida keltirilgan tasdiqga ko'ra

$$t - \tau'_p \leq \sum_{k \geq p+1, \tau_k \leq t} (\tau'_k - \tau_k) \leq \sum_{k \geq p+1, \tau_k \leq t} \frac{4a_k}{\sigma - 1} \leq \frac{4}{\sigma - 1} \sum_{k=p+1}^{\infty} a_k \leq \frac{8a_{p+1}}{\sigma - 1}. \quad (36)$$

Shunday qilib (35) va (36) tengsizliklardan quyidagiga egamiz:

$$\begin{aligned} y_2(t) - x_{p2}(t) &> \frac{1}{2}a_p - \frac{8a_{p+1}}{\sigma-1} > \frac{1}{2}a_p - \frac{a_p}{8\sigma} \\ &> \frac{1}{4}a_p > a_{p+1}, \tau'_p \leq t \leq \bar{\tau}. \end{aligned} \quad (37)$$

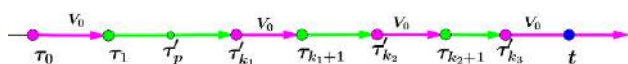
Xulosa qilib aytsek, agar x_p quvuvchi uchun $\tau'_p > \bar{\tau}$ bo'lsa, u holda (31) dan

$$\begin{aligned} |y(t) - x_p(t)| &> a_{p+1}, \\ \tau_p &\leq t \leq \bar{\tau}, \end{aligned} \quad (38)$$

va agar $\tau'_p \leq \bar{\tau}$ bo'lsa, u holda (31) va (37) tengsizliklarni birlashtirib biz (38) tengsizlikni hosil qilamiz. Shuning uchun (38) tengsizlik A va B holarda bir guruh hujumida har bir x_p quvuvchi uchun o'rinli ekan.

(38) tengsizlikdan asosiy xulosa shundan iboratki, $k \geq p+1$ bo'lganda x_p quvuvchi uchun $\tau_p < t \leq \bar{\tau}$ vaqt oralig'ida τ_k yaqinlashish vaqti mavjud. Shunday qilib, bir guruh hujumidagi har bir x_p quvuvchi uchun $\tau_p < t \leq \bar{\tau}$ vaqt oralig'ida faqat bitta τ_p yaqinlashish vaqti mavjud. Natijada, guruh hujumidagi barcha quvuvchilar bir-biridan farq qiladi. Bundan tashqari guruh hujumidagi quvuvchilar soni $\leq m$ bo'ladi, chunki guruh hujumida eng ko'pi bilan m ta yaqinlashish vaqti mavjud. Shuning uchun B holati rad etiladi. Birinchi guruh hujumi $k_1 \leq m$ bo'lgan $x_1, x_2, \dots, x_{k_1}$ quvuvchilar bilan cheklangan sondan iborat. Demak, faqat A holatini ko'rib chiqamiz, bunda $\bar{\tau} = \tau'_{k_1}$ va guruh hujumi τ'_{k_1} da tugaydi.

B. Endi $t \geq \tau'_{k_1}$ uchun (32) tengsizlikni isbotlashga harakat qilamiz, $\tau'_p \leq \tau'_{k_1}$ deb faraz qilamiz. Bu tengsizlik $p = k_1$ (ya'ni x_{k_1} quvuvchi uchun) holatida aniq bajariladi. Endi $t \geq \tau'_{k_1}$ bo'lganda (35) ning o'ng tomonini baholaymiz. Agar $[\tau'_{k_1}, t)$ vaqt oralig'ida bir nechta guruh hujumlari sodir bo'lsa, qochuvchining (i)-(iv) strategiyasiga ko'ra guruh hujumlari orasida va ular tugagandan so'ng $v(t) = V_0$ tezligini saqlab turadi.



3-Rasm . Qochuvchi $[\tau_1, \tau'_{k_1}), [\tau'_{k_1+1}, \tau'_{k_2}), [\tau'_{k_2+1}, \tau'_{k_3})$ intervallarda guruh hujumlari ostida.

3-rasmda $[\tau_1, \tau'_{k_1}), [\tau'_{k_1+1}, \tau'_{k_2}), [\tau'_{k_2+1}, \tau'_{k_3})$ vaqt intervallarida 3 guruh hujumlari bor.

Biz (35) dagi $\int_{\tau'_p}^t v_2(s) ds$ integralni $[\tau'_p, t] = I \cup J, I \cap J = \emptyset$ dan foydalanib baholaymiz, bu yerda $I = [\tau'_p, t]$ oraliqning qochuvchi guruh hujumi ostida bo'lgan qismi, J da esa qochuvchi $v(t) = V_0$ tezlik bilan harakatlanadi. Har bir $t' \in I$ vaqt nuqtasi $[\tau_k, \tau'_k], k \geq p+1$ oraliqga tegishli, shu sababli $I \subset \bigcup_{k \geq p+1} [\tau_k, \tau'_k]$. Binobarin, yuqorida keltirilgan tasdiqdan foydalanib, I ning uzunligini $|I|$ quyidagicha baholanadi :

$$\begin{aligned} |I| &\leq \sum_{k \geq p+1, \tau_k \leq t} (\tau'_k - \tau_k) = \sum_{k \geq p+1, \tau_k \leq t} \frac{4a_k}{\sigma-1} \\ &\leq \frac{4}{\sigma-1} \sum_{k=1}^{\infty} a_k \leq \frac{8a_{p+1}}{\sigma-1}. \end{aligned} \quad (39)$$

Barcha $t \in I$ uchun $v_2(t) > 0$ ligi uchun, $\int_I v_2(s) ds \geq 0$. Bundan va (39) dan foydalanib quyidagiga egamiz:

$$\begin{aligned} \int_{\tau'_p}^t v_2(s) ds &= \int_{I \cup J} v_2(s) ds \\ &= \int_I v_2(s) ds + \int_J \sigma e^{bs} ds \\ &\geq \int_J \sigma e^{bs} ds \\ &= \int_{[\tau'_p, t) \setminus t} \sigma e^{bs} ds \geq \sigma \left(\frac{e^{bt} - e^{b\tau'_p}}{b} - |I| \right) \\ &\geq \sigma \left(\frac{e^{bt} - e^{b\tau'_p}}{b} - \frac{8a_{p+1}}{\sigma-1} \right). \end{aligned}$$

Shuning uchun (35) dan agar $\tau'_p \leq \tau'_{k_1}$ bo'lsa, u holda har qanday $t \geq \tau'_{k_1}$ uchun quyidagi tengsizlik o'rinli:

$$\begin{aligned} y_2(t) - x_{p2}(t) &> \frac{1}{2}a_p - \frac{8\sigma}{\sigma-1}a_{p+1} \\ &\quad + (\sigma-1)(e^{bt} - e^{b\tau'_p}) \\ &\geq \frac{1}{2}a_p - \frac{8\sigma}{\sigma-1}a_{p+1} > \frac{1}{2}a_p - \frac{\varepsilon}{8\sigma}a_p > \frac{1}{4}a_p \\ &> a_{p+1}. \end{aligned}$$

Bu esa $t \geq \bar{\tau}$ vaqt uchun (32) tengsizlikning to'g'ri ekanligini bildiradi. Lemmaning isboti yakunlandi.

Endi, biz **Teoremani** isbotlaymiz.

Isbot. Birinchi guruh hujumidagi x_p quvuvchi uchun $\tau'_p \leq \tau'_{k_1}$ bo'lsin. Hech bo'lmaganda $p = k_1$ bo'lganda bu shart bajariladi. (1) dagi $k = p$ bo'lgan tengsizliklar, (31) va (32) ga ko'ra, barcha $t \geq 0$ vaqtlar uchun $|y(t) - x_p(t)| > a_{p+1}$ bo'ladi. Demak, x_p quvuvchi ikkinchi guruh hujumlaridan boshlab keyingi guruh hujumlarida ishtirok etmaydi, chunki $y_2(t) - x_{p2}(t) > a_{p+1} \geq a_{k_1+1}, t \geq \tau'_p$ tengsizlik o'rinli bo'ladi.

Xuddi shu tarzda agar τ_{k_1+1} vaqt sodir bo'lsa, qochuvchi $[\tau_{k_1+1}, \tau'_{k_2}]$ vaqt oralig'ida ikkinchi guruh hujumi ostida bo'ladi, bu yerda $k_2 \geq k_1 + 1$. Shunday qilib, barcha $t \geq 0$ va biror $q \in \{k_1 + 1, \dots, k_2\}$ uchun $|y(t) - x_q(t)| > a_{q+1}$ bo'ladi, ya'ni x_q quvuvchi uchinchi guruh hujumlaridan boshlab keying hujumlarda ishtirok etmaydi, chunki $y_2(t) - x_{q2}(t) > a_{q+1} \geq a_{k_2+1}, t \geq \tau'_q$ tengsizlik o'rinli bo'ladi va hokazo.

Shunday qilib, dastlabki $x_1, x_2, \dots, x_{k_1}$ quvuvchilarini o'z ichiga olgan guruh hujumidan so'ng, biz kamida bitta quvuvchini (masalan, x_{k_1}) e'tiborsiz qoldirishimiz mumkin. Xuddi shunday

V.Xulosa.

Biz bitta qochuvchi 3 ta quvuvchidan iborat qochish differensial o'yinini o'rgandik. Bu differensial o'yinda quvuvchilar va qochuvchining boshqaruvlari Gronuoll chegaralanishlarga bo'ysunadi. Biz qochuvchi uchun qochish strategiyasini qurdik.

Taklif etilgan strategiyaning asosiy ahamiyati shundaki, guruh hujumida ishtirok etayotgan har bir quvuvchi faqat bir marta yaqinlashish vaqtiga erisha oladi. Shu sababli, birinchi guruh hujumi ko'pi bilan 3 ta yaqinlashish vaqtini, ikkinchisi 2 ta yaqinlashish vaqtini va uchinchisi 1 ta yaqinlashish vaqtini sodir qiladi. Biz yaqinlashish vaqtlarining umumiy soni $3 + 2 + 1 = 6$ dan oshmasligini ko'rsatdik. Taklif qilingan strategiya orqali qochuvchi eng ko'pi bilan 6 manyovr qo'llab, qochishni ta'minlashi mumkinligini isbotladik.

Adabiyotlar

[1] Alias I. A., Ibragimov G., Rakhmanov A. (2016). Evasion Differential Game of Infinitely Many Evaders from Infinitely Many Pursuers in

ikkinchi guruh hujumidan (quvuvchilar $x_{k_1+1}, x_{k_1+2}, \dots, x_{k_2}$) so'ng, biz bu guruhdan yana kamida bitta quvuvchini (masalan, x_{k_2}) hisobga olishimiz shart emas va hokazo. Quvuvchilarning umumiy soni 3 bo'lgani uchun ko'pi bilan 3 ta guruh hujumlaridan so'ng barcha quvuvchilar qochuvchidan "ortda" qoladi. Bu Teoremaning isbotini yakunlaydi.

Endi yaqinlashish vaqtlarining umumiy sonini yuqoridan baholaymiz. Birinchi guruh hujumida ko'pi bilan 3 ta yaqinlashish vaqti bo'lgani uchun, ikkinchi guruh hujumida ko'pi bilan 2 ta yaqinlashish vaqti bo'ladi va hokazo. Shunday qilib, yaqinlashish vaqtlarining umumiy soni ko'pi bilan $3 + 2 + 1 = 6$ ga teng bo'ladi. Agar $\tau_{k_{j_0}}$ oxirgi yaqinlashish vaqti bo'lsa, u holda $k_{j_0} \leq 6$ bo'ladi va hokazo.

$$|x_i(t) - y(t)| > r = a_{k_{j_0}+1}, t \geq 0, i = 1, 2, 3 \quad (47)$$

Bu esa qochuvchi barcha quvuvchilardan kamida r masofani saqlab qochishi mumkinligini anglatadi

Hilbert Space. Dynamic Games and Applications. 10.1007/s13235-016-0196-0, 6(2): 1–13.

[2] Azamov, A.A., Ibaydullayev, T.T. A Pursuit–Evasion Differential Game with Slow Pursuers on the Edge Graph of a Simplex. I. Autom Remote Control 82, 1996–2005 (2021). <https://doi.org/10.1134/S000511792111014X>

[3] Ibragimov, G. Evasion Differential Game of One Evader and Many Slow Pursuers. Dyn Games Appl 14, 665–685 (2024). <https://doi.org/10.1007/s13235-023-00501-2>

[4] Chodun W. (1998). Avoidance of many pursuers in differential games described by differential inclusions. J. Math. Anal. and Appl: 135.

[5] Ibragimov G.I., Ferrara M., Ruziboev M., Pansera B.A. (2021). Linear evasion differential game of one evader and several pursuers with integral constraints. International Journal of Game Theory. 50: 729–750, doi.org/10.1007/s00182-021-00760-6.

[6] Ibragimov G.I., Ferrara M., Kuchkarov A.Sh, and Pansera B.A. (2018). Simple motion

evasion differential game of many pursuers and evaders with integral constraints. *Dynamic Games and Applications*. 8: 352–378. <https://doi.org/10.1007/s13235-017-0226-6>.

[7] Ibragimov G.I., Salleh Y. (2012). Simple motion evasion differential game of many pursuers and one evader with integral constraints on control functions of players. *Journal of Applied Mathematics*, 2012, Article ID 748096, doi:10.1155/2012/748096.

[8] Rilwan, J., Ferrara, M., Badakaya, A.J. et al. On pursuit and evasion game problems with Gronwall-type constraints. *Qual Quant* 57, 5551–5562 (2023). <https://doi.org/10.1007/s11135-023-01616-9>

[9] Rilwan, J., Fotia, P., Ferrara, M. On game value of a differential game problem with Gronwall-type constraints on players control functions. *Decisions Econ Finan* 46, 319–333 (2023). <https://doi.org/10.1007/s10203-023-00389-y>

[10] Samatov B., Akbarov A. (2024). Differential game with a "Life line" under the

Grönwall constraint on controls. *Vestnik of Saint Petersburg University. Applied Mathematics. Computer Science. Control Processes*. 265–280.

[11] Samatov B.T., Ibragimov G.I., Khodjibayeva I.V. (2020) Pursuit-evasion differential games with Grönwall type constraints on controls. *Ural Mathematical Journal*, 6(2): 95–107, DOI: 10.15826/umj.2020.2.010.

[12] Grönwall T.H.(1919) Note on the derivatives with respect to a parameter of the solutions of a system of differential equations. *Ann. Math.* 20(2): 293–296.

[13] Yusupov I.M, Zokirjonov S.M Boshqaruvlari Gronuoll chegaralanishli ikki quvuvchi va bir qochuvchili differensial o‘yin. *NamDU ilmiy axborotnoma* [2025-6]:63-68

Mualliflar haqida ma’lumot

Yusupov Ikrombek Maxammatibroxim o‘g‘li–Andijon davlat universiteti, Amaliy matematika va mexanika kafedrası, o‘qituvchisi. E-mail: yusupovikrom51@gmail.com Tel.: +99891-145-04-

Mathematical models of impurity diffusion in a semi-conductor and their role in electronic technology.

J.R.Alieva

ASU, zhralieva@mail.ru

Abstract: The work analyzes various methods for solving problems related to impurity distribution in a semiconductor after diffusion, corresponding to the chosen mechanism and accepted boundary conditions. Possible ways to use the considered methods for calculating impurity distribution in the creation of instrumental structures and IMC under certain technological conditions are presented.

Keywords: semiconductor, mathematical models of diffusion, Fik's differential equations, IMC (integral microcircuits).

Аннотация: В работе проанализированы различные способы решения задач, относящихся к распределению примесей в полупроводнике после диффузии, соответствующие выбранному механизму и принятым граничным условиям. Представлены возможные пути по использованию рассмотренных методов расчета распределения примесей в создании приборных структур и ИМС при определенных технологических условиях

Ключевые слова: полупроводник, математические модели диффузии, дифференциальные уравнения Фика, ИМС (интегральных микросхем).

High-temperature or low-temperature diffusion of various impurities, including doping impurities, into the volume of a semiconductor is a traditionally developed method for stimulating their electrical conductivity, forming regions of opposite types of conductivity, i.e., electron-hole contacts or even nano-sized volumetric structures [1-2]. To achieve reproducibility of the parameters of the corresponding layers of a semiconductor and, consequently, electronic devices and integrated circuits (IMC), numerous experiments are required, accompanied by inevitable losses of expensive material, energy, etc. An alternative method for selecting optimal technological modes of thermal diffusion of impurities is the preliminary calculation of impurity distribution after thermal diffusion. For performing numerical calculations, it is advisable to derive corresponding equations and their approximate solutions that describe technological processes, taking into account their physical mechanisms and boundary conditions, to which this work is devoted.

For the mathematical description of diffusion processes, Fick's two differential equations are traditionally used. The first equation is written in the form [3-4]:

$$J = -D \text{ grad } N \quad (1)$$

where J - is the density of the diffusing substance flow, i.e., the amount of substance passing through a unit surface area perpendicular to the direction of substance

transfer per unit time; N - impurity atom concentration; D - diffusion coefficient.

The physical meaning of equation (1) is that the primary cause of diffusion mass transfer of a substance is its concentration gradient. The transfer rate is proportional to the concentration gradient, and the diffusion coefficient is introduced as the proportionality coefficient. The minus sign on the right side (1) indicates that diffusion occurs in the direction of decreasing concentration. In other words, diffusion proceeds in the direction of the system reaching physicochemical equilibrium. The process will continue until the chemical potentials of the components of the entire system become equal. Equation (1) describes a stationary process, the parameters of which do not depend on time.

The diffusion coefficient determines the density of the substance flow at a single concentration gradient and is a measure of the rate of concentration gradient equalization. In general, the diffusion is anisotropic, and the diffusion coefficient is a second-rank symmetric tensor.

According to microscopic definition, the D_x component of the diffusion coefficient D along the x -coordinate is related to the root-mean-square displacement of the diffusing atoms along the $\overline{\Delta x^2}$ coordinate and the time interval Δt during which this displacement occurred:

$$D_x = \overline{\Delta x^2} / \Delta t$$

When the concentration of a substance changes only in one direction and diffusion occurs in an isotropic medium, the first Fick equation has the following form:

$$J = -D \frac{\partial N}{\partial x} \quad (2)$$

In the simplest models of doping processes in the technology of manufacturing semiconductor devices and IMC, such diffusion conditions are assumed.

The second equation of diffusion is obtained by combining the first law and the principle of conservation of matter, according to which the change in the concentration of a substance in a given volume must be equal to the difference in the flows of this substance at the inlet and outlet of the volume. In general, the second diffusion equation has the form:

$$\frac{\partial N}{\partial t} = \text{grad}(D \cdot \text{grad}N) \quad (3)$$

For one-dimensional diffusion in an isotropic medium, equation (3) can be written as:

$$\frac{\partial N}{\partial t} = \frac{\partial}{\partial x} \left(D \frac{\partial N}{\partial x} \right) \quad (4)$$

Fick's second law characterizes the process of changing the concentration of diffusing impurities over time at different points of the medium and is a mathematical model of the system's non-stationary state.

When the diffusion coefficient D is constant, equation (4) simplifies:

$$\frac{\partial N}{\partial t} = D \frac{\partial^2 N}{\partial x^2} \quad (5)$$

The assumption of constant diffusion coefficient is valid in a large number of cases analyzed in IMC technology.

Diffusion equations are purely phenomenological, i.e., they contain no

information about the mechanisms of diffusion - about the diffusion process at the atomic level. Equations (1) - (5) do not contain information about the charge state of the diffusing particles.

Diffusion processes used to manufacture IMC are usually analyzed using partial solutions of equation (5) because, unlike (2), it contains an important parameter - the time it takes to establish some analyzed state of the system. The main goal of solving the equation is to find the distribution of impurity $N(x,t)$ in the semiconductor after diffusion for a certain time t under various conditions for the implementation of the process.

The general solution of equation (5) for an infinite solid body with a given initial distribution of impurities $N(x,0) = f(x)$ can be found by the method of separating variables. It has the form:

$$N(x,t) = \frac{1}{2\sqrt{\pi Dt}} \int_{-\infty}^{+\infty} f(\xi) \exp\left[-\frac{(\xi-x)^2}{4Dt}\right] d\xi \quad (6)$$

where ξ is the current integration coordinate.

An infinite in a one-dimensional representation is a body extending from $x = 0$ to $x = -\infty$ and to $x = +\infty$. When searching for the distribution of impurity concentration in a semiconductor, it is necessary to solve equation (5) for a semicontinuous solid. In a one-dimensional representation, a body extending from $x = 0$ to $x = +\infty$ is called semi-infinite. For this case, the expression (6) takes the form:

$$N(x,t) = \frac{1}{2\sqrt{\pi Dt}} \int_{-\infty}^{+\infty} f(\xi) \left\{ \exp\left[-\frac{(\xi-x)^2}{4Dt}\right] \pm \exp\left[-\frac{(\xi+x)^2}{4Dt}\right] \right\} d\xi \quad (7)$$

The plus sign in expression (7) refers to the case when the boundary of a solid body ($x = 0$) is impermeable to the diffusing substance located in the region $x > 0$, and the minus sign refers to the case when the concentration of the

diffusing substance located in the region $x > 0$ at any time on the boundary of a solid body is equal to zero.

The indicated solutions allow us to find the distribution of impurities in a solid body under any initial conditions. Solving a specific problem reduces to substituting the corresponding initial conditions into (6) or (7) with subsequent very cumbersome transformations.

There are various ways to solve equations that take into account the corresponding mechanisms and allow for numerical calculations. It is advisable to analyze the most common of them.

Let's consider the distribution of impurities during diffusion from a semi-infinite space. The diffusing impurity enters a semi-infinite body through the $x = 0$ plane from the second semi-infinite body with uniform impurity distribution. The concentration of impurity in the source is N_0 . It is assumed that there is no considered impurity in the material volume. The initial distribution of concentrations for this case is given as

$$N(x,0) = N_0 \text{ для } x < 0$$

$$N(x,0) = 0 \text{ для } x > 0$$

By solving equation (5) for this case, we can obtain the expression:

$$N(x,t) = \frac{N_0}{2} \left[1 - \frac{2}{\sqrt{\pi}} \int_0^{\frac{x}{2\sqrt{Dt}}} \exp\left(-\frac{y^2}{4Dt}\right) dy \right] \quad (8)$$

The second term in square brackets is called the Gaussian error integral or, in other words, the error function and is abbreviated as $\text{erf}(z)$. According to the abbreviation, this distribution is called the erf distribution.

$$\text{erf } z = \frac{2}{\sqrt{\pi}} \int_0^z \exp(-y^2) dy \quad (9)$$

In mathematics, it is often used as an independent and other function.

$$\text{erfc}(z) = 1 - \text{erf}(z), \quad (10)$$

which is called the addition of the error function to one or the additional error function.

Thus, expression (8) can be written as:

$$N(x,t) = \frac{N_0}{2} \left(1 - \text{erf} \frac{x}{2\sqrt{Dt}} \right) = \frac{N_0}{2} \text{erfc} \frac{x}{2\sqrt{Dt}} \quad (11)$$

The quantity has a length $\sqrt{Dt}$ dimension and is called the diffusion length or diffusion length.

The considered solution can be used as the simplest model representing the distribution of impurities in the autoepitaxial structure. In this case, both the substrate and the epitaxial layer act as independent sources of impurities. In this case, the diffusion processes from each side are considered independent of each other, and the real distribution of impurities at the interface represents the sum of individual solutions.

Let's consider the distribution of impurities during diffusion from a constant source into a semi-infinite body. The diffuser enters the semi-infinite body through the $x = 0$ plane from a source that ensures a constant concentration of impurity N_0 on the solid body-source interface for any time. Such a source is called an infinite or an infinite power source. In this case, it is considered that there is no impurity in the material of the body under consideration. The initial distribution of concentrations and boundary conditions for this case are given in the form

$$N(x,t) = N_0 \text{ для } x = 0$$

$$N(x,0) = 0 \text{ для } x > 0$$

The solution of equation (11) for these conditions is the expression:

$$N(x,t) = N_0 \text{erfc} \frac{x}{2\sqrt{Dt}} \quad (12)$$

If in the volume of a semiconductor material before diffusion there was an impurity of the opposite type to the diffusing one, this impurity was evenly distributed throughout the volume and its concentration was equal to N_b ,

then in this case, an electron-hole junction is formed in the semiconductor. Its position (depth of occurrence) x_j is determined by the condition $N(x, t) = N_b$, from which

$$\frac{N_b}{N_o} = \operatorname{erfc} \frac{x_j}{2\sqrt{Dt}} \quad (13)$$

and

$$x_j = 2\sqrt{Dt} \cdot \operatorname{erfc}^{-1} \left(\frac{N_b}{N_o} \right) \quad (14)$$

here, the notation erfc^{-1} denotes the argument z of the function erfc .

When solving practical problems related to the analysis of diffusion processes, it is necessary to know the amount of impurity Q accumulated in the solid during diffusion over time t . This value is determined by the formula:

$$Q = \int_0^t J(0, t) dt \quad (15)$$

where $J(0, t)$ is the diffusion flux of impurities into the volume through the $x=0$ plane

$$J(0, t) = -D \frac{\partial N}{\partial x} \Big|_{x=0} = \frac{DN_o}{\sqrt{\pi Dt}} \exp \left(-\frac{x^2}{2\sqrt{Dt}} \right) \Big|_{x=0} = N_o \sqrt{\frac{D}{\pi}} \quad \begin{matrix} N(x, 0) = N_o & \text{для } 0 \leq x \leq h \\ N(x, 0) = 0 & \text{для } x > h \end{matrix} \quad (16)$$

from here

$$Q = \int_0^t N_o \sqrt{\frac{D}{\pi}} dt = 2N_o \sqrt{\frac{Dt}{\pi}} \quad (17)$$

Attention should be drawn to the fact that the value of impurities accumulated in the diffusion layer during diffusion with these boundary conditions increases over time.

This model of the diffusion process with a constant source describes the process of diffusion doping of a semiconductor material from a gas or vapor phase. This process is used to create highly doped diffusion layers with surface concentrations N_o , close to the values of the impurity's maximum solubility in a given semiconductor material.

A solid body can be considered semi-finite (or infinite) if its dimensions in the direction of diffusant motion are much greater than the diffusion length.

Let's consider the distribution of impurities during diffusion from a finite thickness layer into a semi-infinite body with a reflecting boundary. The diffusing impurity enters the semi-infinite body from a source, which is a layer of thickness h adjacent to the body's boundary, in which the impurity is evenly distributed. Such a source is called a limited source, and the impurity concentration in it is N_o . In this case, it is assumed that there is no impurity in the material under consideration.

At a boundary absolutely impermeable to the diffuser, the impurity flux through the surface $x = 0$ must become zero for all $t \geq 0$

$$\frac{\partial N(x, t)}{\partial x} \Big|_{x=0} = 0 \quad \text{for } t \geq 0 \quad (18)$$

The initial distribution of concentrations for the case under consideration is given as

The boundary condition, determined by condition (18), is the constant amount of impurity in the source and semiconductor

$$\int_0^{\infty} N(x, t) dx = Q$$

To implement this type of initial distribution, the diffusing impurity must be introduced into the semiconductor before diffusion begins.

The solution of equation (11) in this case is the expression

$$N(x, t) = \frac{N_o}{2} \left(\operatorname{erfc} \frac{h-x}{2\sqrt{Dt}} + \operatorname{erfc} \frac{h+x}{2\sqrt{Dt}} \right) \quad (19)$$

Unlike diffusion from a constant source, diffusion from a finite thickness layer limits the amount of diffusing impurities to the value $Q=N_0h$. During diffusion, only its redistribution occurs, and consequently, the concentration of impurities on the solid surface decreases over time.

An example of impurity diffusion from a finite thickness layer into a semi-infinite body with a reflecting boundary is diffusion into a silicon plate from an epitaxial, implanted, or diffusion layer and coated with a layer of silicon dioxide SiO_2 or silicon nitride Si_3N_4 . The boundary between the plate and the film can be assumed to be reflective with a high degree of truth, since the diffusion coefficients of most impurities in silicon are several orders of magnitude greater than in silicon dioxide and nitride. However, the uniform distribution of impurity in the source, especially when it is created by diffusion or implantation, is a rather crude and forced approximation.

Let us consider the distribution of impurities during diffusion from an infinitely thin layer into a semi-infinite body with a reflecting boundary. The solution of the diffusion equation under these conditions is found from the previous one for $h \rightarrow 0$ and the condition that the number of diffusing impurities in the source $Q=N_0h$.

$$N(x,t) = \frac{Q}{\sqrt{\pi Dt}} \exp\left(-\frac{x^2}{4Dt}\right) \quad (20)$$

The given expression represents a Gaussian distribution.

During diffusion, the thin layer on the surface of the semiconductor plate is quickly depleted, and a constant decrease in the surface concentration of impurities in the semiconductor is observed. This feature of this process is used in semiconductor technology to obtain controlled values of low surface concentration of impurities, for example, to create base regions of silicon transistor structures of discrete devices or IMC.

In the first stage of the process, short-term low-temperature diffusion is carried out

from a constant source, after which the distribution of impurities is described by expression (8). The value of N_0 is large and is mainly determined by the solubility limit of this impurity in the semiconductor. This stage is often called distilling. After the completion of the first stage, the plates are placed in another furnace for further diffusion, usually at higher temperatures. Thus, the thin layer obtained in the first stage is the source of the redistributed impurity during the second stage of the process. To create a reflective boundary, the second stage, called distillation, is carried out in an oxidizing atmosphere, where a layer of SiO_2 grows on the surface.

In practice, a noticeable discrepancy is observed between the distribution of impurities in the source formed during distillation and the distribution derived from expressions (19) and (20) - stepwise. The discrepancy should be reflected in the accuracy of the description of the real distribution of impurities after the second stage of diffusion by expression (20).

When modeling two-stage diffusion and analyzing the process results, it is assumed that the expression (20) sufficiently accurately corresponds to the real one, provided that the value of the D_1t_1 product for the first stage of the doping process is significantly less than D_2t_2 for the second, i.e., $(D_2t_2 / D_1t_1) \geq 100$. This is the condition for the rapid depletion of the source. In this case, considering that the amount of impurity accumulated at the first stage is determined by the ratio

$$Q = 2N_o \sqrt{\frac{D_1t_1}{\pi}}$$

from (20) we obtain:

$$N(x,t) = \frac{2N_o}{\pi} \sqrt{\frac{D_1t_1}{D_2t_2}} \exp\left(-\frac{x^2}{4D_2t_2}\right) \quad (21)$$

If the duration of the second stage is not very long compared to the first, or in other words, $D_2t_2 \ll D_1t_1$, the assumption that the diffusion layer formed as a result of distillation will behave as a thin source is incorrect. In this case, the solution of the diffusion equation will look like this.

$$N(x, t_1, t_2) = N_0 \frac{2}{\pi} \int_b^{\infty} \exp(-m^2) \cdot \operatorname{erf}(a \cdot m) dm \quad (22)$$

where

$$a = \sqrt{\frac{D_1 t_1}{D_2 t_2}} \quad \text{и} \quad b = \frac{x}{2\sqrt{D_1 t_1 + D_2 t_2}}.$$

The surface concentration of impurities after the second stage of diffusion under given conditions is expressed by the ratio

$$N_s = \frac{2N_0}{\pi} \operatorname{arctg} \sqrt{\frac{D_1 t_1}{D_2 t_2}} \quad (23)$$

The expression (19) is used to represent the distribution under the condition that $D_1 t_1 > D_2 t_2$, namely - $(D_2 t_2 / D_1 t_1) < 0,06$. At the same time, it is believed that $h = \sqrt{D_1 t_1}$.

Thus, various methods for solving impurity distribution problems in a semiconductor after diffusion, corresponding to the chosen mechanism and boundary conditions, were analyzed. To obtain the quantitative distribution of impurities, appropriate computer calculations are required using equations (8), (11), (12), (19) or (20) - (23) with the introduction of physical parameters of the material and parameters of the selected technological modes. Possible ways to use the considered methods for calculating impurity distribution in the creation of instrumental structures and IMC under certain technological conditions are shown.

Literature

- [1]. Barybin, A.A. Electronics and Microelectronics. Physical and technological foundations. M.: Fizmatlit, 2006. - 424 p.
- [2]. M.K.Bakhadyrkhanov, G.Kh.Mavlonov, S.B.Isamov, Kh.M.Iliev, K.S.Ayupov, Z.M.Saparniyazova, S.A.Tachilin. Electrophysical properties of silicon doped with manganese by low-temperature diffusion. // Inorganic Materials, 2011, Volume 47, No. 545-550.
- [3]. Boltax, B.I. Diffusion and Point Defects in Semiconductors. - L.: Nauka, 1972. - 384 p.
- [4]. Kolobov, N.A. Diffusion and Oxidation of Semiconductors. M.: Metallurgy, 1975. - 456 p.

UDK 538.9 : 669.14 : 544.35

Cost-effective synthesis of graphene dispersions by liquid-phase exfoliation and their structural evaluation

R.Sh.Daminov¹, J.K.Avloni², A. Turayev², M.S. Qambarova²

^{1,2} Nanotechnology Development Center, National University of Uzbekistan named after Mirzo Ulugbek. Tashkent 100174, e-mail: daminov_r@nuu.uz

Аннотация

В данной работе исследован метод жидкофазной эксфолиации (Liquid-Phase Exfoliation, LPE) для получения дисперсий графена из графита. Для диспергирования графита использовался раствор карбоксиметилцеллюлозы (КМЦ), при этом применялось высокосдвиговое перемешивание. Полученные образцы были проанализированы с использованием спектроскопии Рамана и UV спектроскопии. Результаты UV анализа показали наличие максимума поглощения в диапазоне ~270–280 нм, что соответствует электронным переходам $\pi \rightarrow \pi^*$ и подтверждает образование дисперсий графена. Спектры Рамана с наличием пиков D, G и 2D подтвердили формирование структуры графена, а также указали на увеличение количества дефектов в процессе эксфолиации. Проведённые исследования подтвердили, что метод LPE является простым, экономичным и экологически безопасным способом приготовления дисперсий графена.

Ключевые слова: графен, графит, жидкофазная эксфолиация, высокосдвиговое перемешивание, карбоксиметилцеллюлоза (КМЦ), рамановская спектроскопия, структурные свойства, оптические свойства, экономически эффективный синтез.

Abstract

This study investigates the liquid-phase exfoliation (LPE) method for producing graphene dispersions from graphite. Carboxymethylcellulose (CMC) solution was used to facilitate graphite dispersion, and high-shear mixing was applied. The obtained samples were analyzed using Raman and UV-Vis spectroscopy. UV-Vis results revealed a maximum absorption peak in the ~270–280 nm range, confirming $\pi \rightarrow \pi^*$ electronic transitions and the formation of graphene dispersions. Raman spectra, with the presence of D, G, and 2D peaks, confirmed the formation of a graphene structure and indicated an increase in defects during the exfoliation process. The results demonstrate that LPE provides a simple, cost-effective, and environmentally friendly approach for preparing graphene dispersions.

Keywords: Graphene, Graphite, Liquid-Phase Exfoliation, High-Shear Mixing, Carboxymethylcellulose (CMC), Raman Spectroscopy, Structural Properties, Optical Properties, Cost-Effective Synthesis

I. Introduction.

Graphene is a two-dimensional nanomaterial composed of carbon atoms arranged in a single layer of sp^2 -hybridized atoms forming a honeycomb lattice. Since its first experimental isolation in 2004, graphene has been recognized as one of the most promising materials in the field of materials science [1,2]. Its unique physicochemical properties—including an exceptionally high electron mobility of $2.5 \times 10^5 \text{ cm}^2/\text{V}\cdot\text{s}$, outstanding thermal conductivity of $3000 \text{ W}\cdot\text{m}/\text{K}$, remarkable mechanical strength with a Young's modulus of 1 TPa, optical transparency of 97.4%, and a theoretically large surface area of $\sim 2600 \text{ m}^2/\text{g}$ —make it a key material for modern technologies [3,4].

Such multifunctional properties of graphene have spurred extensive research into its potential applications in a wide range of advanced

fields, including flexible electronics [5], supercapacitors and batteries [6,7], transparent electrodes [8,9], highly sensitive sensors, structural nanocomposites [10,11], water purification systems, and biomedical devices [12,13]. Consequently, scientific and industrial communities worldwide have launched large-scale initiatives to accelerate the commercialization of graphene-based products. The European Union's "Graphene Flagship" project is a notable example.

Nevertheless, the large-scale industrial implementation of graphene still faces several challenges. Chief among these is the lack of cost-effective, environmentally friendly, and scalable production technologies. Traditional "bottom-up" approaches (such as chemical vapor deposition) enable the synthesis of high-quality graphene but are costly, complex, and equipment-intensive.

In contrast, “top-down” approaches—based on exfoliating graphene layers from graphite—stand out for their simplicity, affordability, and scalability. Therefore, this article explores the scientific foundations of contemporary top-down graphene production techniques, their advantages and limitations, and their industrial prospects through experimental studies and analytical assessments.

II. Materials and methods.

The existing technologies for graphene production are generally divided into two main approaches: *top-down* and *bottom-up*. Both approaches allow the fabrication of materials suitable for different applications while preserving the physicochemical properties of graphene; however, they differ in technological methodology, complexity, and cost-effectiveness. The *bottom-up* synthesis methods of graphene are based on constructing graphene molecules from constituent compounds, namely carbon precursors. In this approach, graphene is formed through chemical reactions of carbon sources. The principal advantage of bottom-up methods lies in their precise control over structure and the number of layers, which enables the production of high-quality graphene with minimal defects. The most widely used bottom-up methods include: Chemical Vapor Deposition (CVD), thermal pyrolysis, epitaxial growth, laser-assisted synthesis, and organic synthesis [14].

Among them, the CVD (Chemical Vapor Deposition) method is the most prominent bottom-up technique. In this process, various carbon-containing gases (e.g., CH_4 , C_2H_2) decompose on metal substrates (e.g., Ni, Cu), forming graphene layers. CVD technology enables the production of high-quality single-layer graphene (SLG) or few-layer graphene (FLG) [15]. However, the process requires high temperatures, vacuum systems, and sophisticated equipment, making it economically costly.

The epitaxial growth method involves producing graphene layers on SiC substrates at high temperatures. Although this method allows the synthesis of high-quality graphene, it is expensive and technologically demanding on an industrial scale.

Thermal pyrolysis entails heating various carbon polymers or organic compounds to generate graphene. While this method allows the production of functionalized graphene at relatively low temperatures, the resulting structures often contain numerous defects.

In the laser-assisted synthesis process, laser irradiation is applied to graphene oxide (GO) or polymers, enabling the rapid formation of graphene. This method is simple, environmentally benign, and allows the fabrication of graphene with specific shapes and structures. In particular, porous graphene produced by laser techniques is highly advantageous in applications requiring a large surface area, such as supercapacitors and catalysis.

Finally, organic synthesis constructs graphene structures through molecular assembly from small molecules. This technique enables the production of nanographene or graphene quantum dots (GQDs) with high precision and controlled edge structures. However, due to its multi-step and expensive nature, this method is unsuitable for large-scale production.

In general, although the bottom-up approach allows the production of high-quality graphene, these methods are complex, expensive, and time-consuming for large-scale manufacturing.

The *top-down* approach, by contrast, is based on exfoliating graphene from multilayer carbon materials such as graphite and is currently one of the most widely used methods at the industrial scale. These methods aim to obtain single- or few-layer graphene through exfoliation and reduction of the multilayer graphite structure by mechanical, chemical, or electrochemical processes. The important advantages of top-down methods include their simplicity, low cost, and suitability for large-scale production. The following main methods are widely used in top-down synthesis.

Mechanical exfoliation. In this method, micrometer-sized graphene layers are separated from graphite crystals using ordinary adhesive tape (Scotch tape). This method was first employed by Novoselov and Geim in 2004,

which led to their Nobel Prize [1]. Although this method allows obtaining very high-quality monolayer graphene, it is considered impractical for practical applications due to its limited production scale. Liquid-phase exfoliation (LPE). In the LPE method, the interlayer distance of graphite is expanded with the help of chemical substances, and then the layers are separated by ultrasonication, high-speed stirring, or rapid heating. As a result, few-layer or even monolayer graphene can be obtained. This method is convenient for industry; however, the resulting graphene particles are usually small and structurally defective.

Electrochemical exfoliation. In this method, graphite is used as an electrode, and ions are inserted into the graphite layers with the help of electric potential. As the ions generate gas and expand inside the graphite, the layers separate from each other. Although this method is environmentally friendly, fast, and simple, there are problems in controlling graphene quality. **Oxidation–Reduction.** This is the most common top-down method. The process begins with converting graphite into graphite oxide using strong oxidizing agents. Then, through sonication, graphite oxide is transformed into graphene oxide (GO), and finally, GO is reduced with reducing agents to form reduced graphene oxide (rGO). Although rGO has electrical conductivity, it may still contain residual oxygen groups and defects in its structure. This method is inexpensive and suitable for mass production but is not perfect in terms of quality.

The general disadvantages of top-down methods are the defective structure of graphene, the difficulty of precise control over the number of layers, and the presence of environmental risks in some techniques. Nevertheless, this approach is currently considered one of the most promising and widely used methods for large-scale production. Therefore, although top-down methods are more often chosen in practice, bottom-up methods are advantageous in areas where high-quality graphene is scientifically required.

LPE (Liquid Phase Exfoliation) includes several approaches that directly utilize a liquid

medium to exfoliate synthetic or natural graphite into monolayer or few-layer graphene. Since the first report in 2008 by Coleman and colleagues on exfoliating graphite using sonication, multiple LPE approaches—such as microfluidization, jet cavitation, high-shear mixing, and high-pressure homogenization—have been extensively studied. Each approach has its own advantages and limitations. One of the positive aspects of these methods is that they do not require a complex chemical oxidation step, are easy to control, and do not demand sophisticated equipment, high temperatures, or vacuum conditions. Therefore, the LPE method is considered a promising option for the inexpensive and large-scale production of graphene [16,17].

Although LPE methods generally produce graphene of moderate quality, the material remains valuable because defects can be beneficial in certain applications. For example, edge defects in graphene make it useful for catalysis and supercapacitor applications, while porous graphene structures are suitable for semiconductor devices that require a bandgap.

Another important factor influencing the properties of graphene obtained via LPE is the medium and technique used for exfoliation. To date, a variety of media—including organic solvents, water, surfactant/water solutions, ionic liquids, and aromatic solvents—have been investigated.

At the same time, microfluidization and high-shear mixing are being recognized as promising and novel LPE approaches for exfoliating graphene into layers. Our study focuses on results obtained from high-shear mixing, which have been examined and discussed.

III. Experimental Section

The materials and equipment used were as follows: carboxymethylcellulose obtained from “Tashkent Polymer Synthesis”; graphite powder “Graphite ГЛ-1” supplied by “Тайгинский горно-обогатительный комбинат”; DI water; UV-Vis spectrophotometer “SHIMADZU 1900”; blender “VITAMIX 5200”; Raman spectrometer “RENISHAW”; and electromagnetic stirrer “DLAB MS-H280 Pro.”

For the purpose of exfoliating graphene from graphite in a liquid phase, a solution was first prepared. To prevent the exfoliated graphene flakes from re-aggregating and to ensure dispersion in the solution, “CMC” was used, while “Graphite ГГI-1” powder served as the source material for graphene exfoliation.

Initially, 600 ml of a 0.5% CMC solution was prepared. After 6 hours, 400 ml of DI water was added to reduce the concentration and dilute the solution, followed by another 6 hours of stirring with the electromagnetic stirrer. After 12 hours, the solution was transferred into the high-speed blender “VITAMIX 5200.” At 1000 rpm, 100 g of graphite powder was gradually added and stirred for 5 minutes. In the next stage, the mixture was stirred at 5000 rpm for 10 minutes. To ensure that the temperature did not exceed 50 °C, the solution was stored in a refrigerator for 30 minutes after every 10 minutes of stirring, and then the process was resumed.

Samples were collected every 100 minutes. Each sample was subjected to centrifugation at 4000 rpm for 5 minutes to separate the upper (supernatant) and lower (sediment) fractions. The experiment lasted for 1000 minutes. As a result, samples stirred for 100, 300, 400, and 1000 minutes were obtained and analyzed.

IV. Results and Discussion

To confirm the formation of graphene in the obtained sample, Raman spectral analysis was carried out. Figure 1 shows the Raman spectra of the synthesized graphene sample. Using Raman spectroscopy, the structural properties of the prepared graphene-based sample were analyzed.

The spectrum revealed three main peaks located around 1350 cm^{-1} (D band), 1580 cm^{-1} (G band), and 2700 cm^{-1} (2D band). The G peak corresponds to the E_{2g} phonon vibration of sp^2 -bonded carbon atoms, indicating the presence of a graphite-like structure. The D peak corresponds to the “breathing” vibrations of sp^2 carbon rings and becomes active only due to structural defects or edge sites, suggesting that the graphene structure contains a certain degree of disorder. The 2D peak is the second-order harmonic of the D peak and provides important information about

the number of graphene layers. The relatively sharp and symmetric shape of the 2D peak, along with its moderate intensity compared to the G peak, indicates the formation of few-layer graphene (FLG) [18].

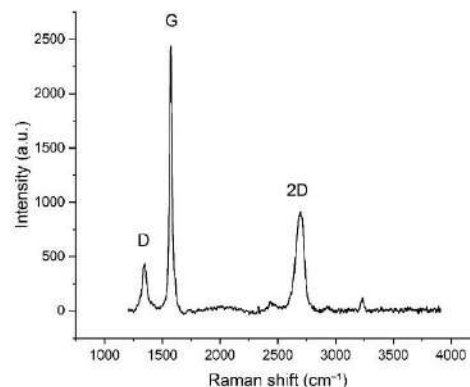


Figure 1. Raman spectrum of the sample obtained after 500 minutes of mechanical exfoliation.

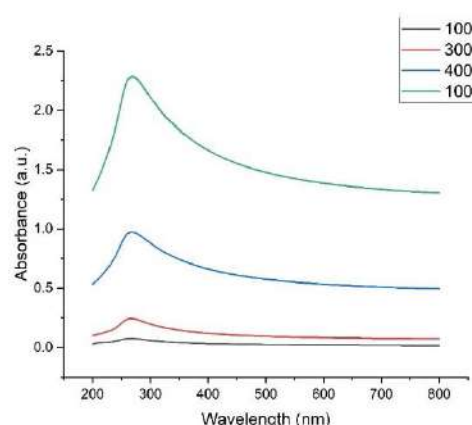


Figure 2. UV-Vis spectroscopy analysis of the samples obtained after 100, 300, 400, and 1000 minutes of mechanical exfoliation.

The UV-Vis spectra of the samples are presented in Figure 2. According to the UV-Vis spectroscopy results, the samples exhibited a maximum absorption peak around 270–280 nm, which corresponds to $\pi \rightarrow \pi^*$ electronic transitions, indicating the presence of a graphene structure [19]. The absorption coefficient increased with stirring time (from 100 to 1000 minutes), reflecting enhanced dispersion efficiency or an increase in the concentration of

exfoliated graphene. Notably, the highest absorption was observed after 1000 minutes of exfoliation, indicating the presence of the largest amount of exfoliated graphene particles.

V. Conclusion.

In this study, a graphene dispersion was prepared by adding 100 g of graphite to a 0.3% CMC solution and stirring at 5000 rpm for 1000 minutes. Samples were collected at 100-minute intervals, and their structural and optical properties were analyzed using Raman and UV-Vis spectroscopy.

Based on UV-Vis spectral analysis, all samples showed a maximum absorption peak around $\sim 270\text{--}280\text{ nm}$, corresponding to $\pi \rightarrow \pi^*$ electronic transitions and indicating the presence of a graphite/graphene structure. With increasing stirring time, the absorption coefficient significantly increased, demonstrating that more graphene particles were produced in the dispersion. In particular, the sample stirred for 1000 minutes reached an absorption coefficient of 2.5, showing the highest degree of exfoliation.

The Raman spectra provided information about the crystallinity and defect level of the material. Analysis revealed that all samples exhibited D ($\sim 1350\text{ cm}^{-1}$), G ($\sim 1580\text{ cm}^{-1}$), and 2D ($\sim 2700\text{ cm}^{-1}$) peaks. The presence of the G peak is associated with sp^2 -bonded carbon atoms, while the D peak indicates structural defects and edge sites of graphene flakes. The relatively sharp and intense 2D peak suggests the presence of few-layer graphene in the material. The increase in the D peak intensity with longer stirring times indicates a rise in defect concentration, confirming that an effective exfoliation process took place.

Overall, the conducted research demonstrates that graphene can be obtained in a simple, cost-effective, and efficient manner via liquid-phase exfoliation using CMC to disperse graphite. This approach may serve as a promising method for preparing stable, water-based, and environmentally friendly graphene dispersions in the future.

Reference

- [1] K.S. Novoselov, A.K. Geim, S.V. Morozov, D. Jiang, Y. Zhang, S.V. Dubonos, I. V. Grigorieva, A.A. Firsov, *Science* 306 (2004) 666–669.
- [2] A.K. Geim, K.S. Novoselov, The rise of graphene, in: *Nanoscience and technology: a collection of reviews from nature journals*, World Scientific, 2010, pp. 11–19.
- [3] K.S. Novoselov, V.I. Fal'ko, L. Colombo, P.R. Gellert, M.G. Schwab, K. Kim, *Nature* 490 (2012) 192–200.
- [4] S. Gadipelli, Z.X. Guo, *Prog. Mater. Sci.* 69 (2015) 1–60. [6] Y. Cao, V. Fatemi, S. Fang, K. Watanabe, T. Taniguchi, E.
- [5] S. Bae, H. Kim, Y. Lee, X. Xu, J.-S. Park, Y. Zheng, J. Balakrishnan, T. Lei, H. Ri Kim, Y.I. Song, Y.-J. Kim, K.S. Kim, B. "Ozyilmaz, J.-H. Ahn, B.H. Hong, S. Iijima, *Nat. Nanotechnol.* 5 (2010) 574–578.
- [6] W. Yang, M. Ni, X. Ren, Y. Tian, N. Li, Y. Su, X. Zhang, *Curr. Opin. Colloid Interface Sci.* 20 (2015) 416–428.
- [7] Tsai, J. Cao, L. Le Fevre, B. Wang, R. Todd, R.A. Dryfe, A.J. Forsyth, *Electrochim. Acta* 257 (2017) 372–379.
- [14] J. Phiri, P. Gane, T.C. Maloney, *Mater. Sci.*
- [8] R.K.L. Tan, S.P. Reeves, N. Hashemi, D.G. Thomas, E. Kavak, R. Montazami, N. N. Hashemi, *J. Mater. Chem. A* 5 (2017) 17777–17803.
- [9] M.M. Tavakoli, G. Azzellino, M. Hempel, A.-Y. Lu, F.J. Martin
- [10] J. Phiri, P. Gane, T.C. Maloney, *Mater. Sci. Eng., B* 215 (2017) 9–28.
- [11] K. Hu, D.D. Kulkarni, I. Choi, V.V. Tsukruk, *Prog. Polym. Sci.* 39 (2014) 1934–1972. [16] R.K.L. Tan, S.
- [12] R. Gusain, N. Kumar, S.S. Ray, *Coord. Chem. Rev.* 405 (2020), 213111
- [13] J. Shi, Y. Fang, *Biomedical Applications of Graphene*, in: *Graphene*, Elsevier, 2018, pp. 215–232.
- [14] M.S.A. Bhuyan, M.N. Uddin, M.M. Islam, F.A. Bipasha, S.S. Hossain, *International Nano Letters* 6 (2016) 65–83.
- [15] J.Y. Lim, N. Mubarak, E. Abdullah, S. Nizamuddin, M. Khalid, *J. Ind. Eng. Chem.* 66 (2018) 29–44.
- [16] Y. Xu, H. Cao, Y. Xue, B. Li, W. Cai, *Nanomaterials* 8 (2018) 942.
- [17] K.R. Paton, E. Varilla, C. Backes, R.J. Smith, U. Khan, A. O'Neill, C. Boland, M. Lotya, O.M. Istrate, P. King, T. Higgins, S. Barwich, P. May, P. Puczkarski, I. Ahmed, M. Moebius, H. Pettersson, E. Long, J. Coelho, S.E. O'Brien, E. K. McGuire, B.M. Sanchez, G.S. Duesberg, N. McEvoy, T.J. Pennycook, C. Downing, A. Crossley, V. Nicolosi, J.N. Coleman, *Nat. Mater.* 13 (2014) 624–630.
- [18] Ajith Pattammattel and Challa Vijaya Kumar. *Kitchen Chemistry 101: Multigram Production of High Quality Biographene in a Blender with Edible Proteins advanced functional material* 2015 on.p. 7088 7098.
- [19] Min Yi a,b,* , Zhigang Shen. *Kitchen blender for producing high-quality few-layer graphene* CARBON 78 o.p. 622-626

Guaranteed evasion time in a differential game for an infinite system in hilbert space l_2

¹⁾Dilnoza Uktamjonovna Kushakova

²⁾Tulanboy Tursunbaevich Ibaydullaev

dilnozaqoshaqova717@gmail.com

ibaydullayev73@mail.ru

Abstract. This study examines an evasion-type differential game governed by an infinite system of binary differential equations formulated in a Hilbert space. The control functions of the players are subject to geometric constraints. The pursuer aims to drive the state of the system toward the origin, whereas the evader's objective is to prevent this. An analytical expression for the guaranteed evasion time is rigorously derived.

Keywords: differential game, Hilbert space, evasion strategy, infinite-dimensional systems, guaranteed evasion time, geometric constraints.

Аннотация. В данной работе исследуется дифференциальная игра типа уклонения, описываемая бесконечной системой бинарных дифференциальных уравнений, сформулированных в гильбертовом пространстве. Управляющие функции игроков подчинены геометрическим ограничениям. Преследователь стремится привести состояние системы к началу координат, в то время как цель уклоняющегося состоит в том, чтобы предотвратить это. Строго выводится аналитическое выражение для гарантированного времени уклонения.

Ключевые слова: дифференциальная игра, стратегия уклонения, Гильбертово пространство, бесконечномерные системы, гарантированное время уклонения, геометрические ограничения.

Annotatsiya. Ushbu maqolada Gilbert fazosida cheksiz ikkitalik differensial tenglamalar sistemasi bilan boshqariladigan qochish turidagi differentsial o'yin o'rganiladi. O'yinchilarning boshqaruv funksiyalari geometrik cheklovlarga ega bo'lib, quvlovchi sistema holatini koordinata boshiga olib borishga harakat qiladi, qochuvchining maqsadi esa bunga to'sqinlik qilishdir. Kafolatli qochish vaqtining analitik ifodasi qat'iy ravishda keltirib chiqariladi.

Kalit so'zlar: differentsial o'yin, Gilbert fazosi, qochish strategiyasi, cheksiz o'lchovli sistemalar, kafolatli qochish vaqti, geometrik cheklovlar.

1. INTRODUCTION

Differential games constrained by partial differential equations (PDEs) have emerged as a significant area of research in control theory. Initial advancements were made by Fattorini through his investigation of time-optimal control in parabolic-type PDEs. Subsequent foundational contributions by Lions and Osipov extended these concepts to differential games. Liu et al. further examined optimal control problems on domains with complex geometry, while Geshkovski and Zuazua introduced an analytical framework for establishing turnpike properties in linear PDE-constrained optimization problems.

The decomposition method has proven instrumental in transforming PDE problems into systems of ordinary differential equations (ODEs), facilitating their analysis. This methodology has been employed in various works by researchers such as Butkovskiy, Chernous'ko, Avdonin, Ivanov, Satimov, Tukhtasinov, Chaves-Silva, Alimov,

Albeverio, and Martin. Despite the apparent simplification, challenges persist due to the inherently infinite-dimensional nature of the resulting ODE systems.

Also, using this method, some differential game problems governed by PDE were analyzed (see, for example, [3, 12, 19, 20, 22, 23]). Though this method results in a differential game described by a system of ordinary differential equations, the main difficulty in solving differential game problems arises because of the infinity of the number of equations in the system.

In such contexts, a PDE may equivalently be represented by an infinite system of the form:

$$\frac{\partial z}{\partial t} = Az + u - v,$$

$$Az = - \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial z}{\partial x_j} \right). \quad (1)$$

These differential game problems described by (1) were reduced to a differential game described by the following infinite system of differential equations

$$\dot{z}_k + \lambda_k z_k = u_k - v_k, \quad k = 1, 2, \dots, \quad (2)$$

where u_k and v_k are control parameters of players, $z_k, u_k, v_k \in \mathbb{R}$, and the coefficients $\lambda_k, k = 1, 2, \dots$, satisfy the condition $0 < \lambda_1 \leq \lambda_2 \leq \dots \rightarrow \infty$.

Therefore, assuming that $\lambda_1, \lambda_2, \dots$ are any numbers, differential games for the infinite system of differential equations (2) can be studied separately from PDE (1). In its turn, infinite system of differential equations (2) can be extended by replacing the coefficients λ_k with $d \times d$ matrices A_k , and z_k, u_k, v_k with the d -dimensional vectors. For example, the work studies the problem of optimal pursuit for the following infinite system:

$$\begin{aligned} \dot{x}_i &= -\alpha_i x_i - \beta_i y_i + u_{i1} - v_{i1}, & x_i(0) &= x_{i0}, \\ \dot{y}_i &= \beta_i x_i - \alpha_i y_i + u_{i2} - v_{i2}, & y_i(0) &= y_{i0}, \end{aligned}$$

in Hilbert space l_2 , where α_i, β_i are real numbers, and $\alpha_i \geq 0$. In fact, system (3) can be derived from system (2) when λ_k are complex numbers and z_k, u_k, v_k are complex variables. Also, by replacing $\dot{x}_i, \dot{y}_i$ with the second derivatives $\ddot{x}_i, \ddot{y}_i$ in system (3) we can study a differential game described by the infinite system of second order binary equations (see, for example, [10]). Similar to finite dimensional differential games one can consider differential games with geometric or integral constraints (see, for example, [17]), multi pursuer differential games (see, for example, [18]) and so on for the infinite systems of differential equations.

In the work a differential game with geometric constraints for equation (2) was considered and an equation for the guaranteed pursuit time was obtained. The paper is devoted to a differential game for system (3) with geometric constraints, and an equation for the guaranteed pursuit time was obtained.

Infinite systems of equations (2) and (3) can be written as an infinite system of 2-systems as follows:

$$\dot{z}_i = A_i z_i + u_i - v_i, \quad A_i = \begin{bmatrix} -\lambda_{i1} & 0 \\ 0 & -\lambda_{i2} \end{bmatrix},$$

$$\dot{z}_i = B_i z_i + u_i - v_i, \quad B_i = \begin{bmatrix} -\alpha_i & -\beta_i \\ \beta_i & -\alpha_i \end{bmatrix},$$

where $\lambda_{i1}, \lambda_{i2}, \alpha_i, \beta_i$ are some real numbers, $z_i = (x_i, y_i)$, $u_i = (u_{i1}, u_{i2})$, $v_i = (v_{i1}, v_{i2})$. Clearly, the matrices A_i and B_i are the Jordan matrices. Any 2x2 real matrix is similar to one of the Jordan matrices:

$$A_i, B_i \text{ or } C_i = \begin{bmatrix} -\lambda_i & 1 \\ 0 & -\lambda_i \end{bmatrix},$$

with λ_i being a real number. For the completeness, equations (2) and (3) suggests considering an infinite system of 2-equations corresponding to matrices C_i . Therefore, in the present paper, we study a differential game problem described by the following infinite system of binary differential equations

$$\dot{z}_i = C_i z_i + u_i - v_i, \quad i = 1, 2, \dots$$

For this system, neither a differential game with (3) integral constraints nor a differential game with geometric constraints was studied in the literature. We study a differential game with geometric constraints, i.e., the values of the control functions of pursuer and evader belong to the balls of radii ρ and σ , respectively. We obtain an equation for the guaranteed pursuit time. Also, we obtain a formula for a guaranteed evasion time.

2. MATERIALS AND METHODS

We recall that the Hilbert space l_2 is a linear space of all sequences of real numbers

$$l_2 = \left\{ \xi = (\xi_1, \xi_2, \dots, \xi_n, \dots) \mid \sum_{n=1}^{\infty} \xi_n^2 < \infty \right\}$$

where the inner product and norm are defined as follows

$$(\xi, \eta) = \sum_{n=1}^{\infty} \xi_n \eta_n, \quad \|\xi\| = \sqrt{(\xi, \xi)}.$$

A controlled object is described by the following infinite system of differential equations

$$\begin{aligned} \dot{x}_i &= -\lambda_i x_i + y_i + u_{i1} - v_{i1}, & x_i(0) &= x_{i0}, \\ \dot{y}_i &= -\lambda_i y_i + u_{i2} - v_{i2}, & y_i(0) &= y_{i0}, \quad i = 1, 2, \dots, \end{aligned} \quad (4)$$

in Hilbert space l_2 , where $\lambda_i \geq \lambda_0$, and λ_0 is a given positive number, $x_0 = (x_{10}, x_{20}, \dots) \in l_2$, $y_0 = (y_{10}, y_{20}, \dots) \in l_2$, $u = (u_{11}, u_{12}, u_{21}, u_{22}, \dots)$ and $v = (v_{11}, v_{12}, v_{21}, v_{22}, \dots)$ are the control parameters of pursuer and evader, respectively. We suppose that $0 \leq t \leq T$, where T is a sufficiently large number, and that $\eta_0 = (\eta_{10}, \eta_{20}, \dots) = (x_{10}, y_{10}, x_{20}, y_{20}, \dots) \neq 0$, where $\eta_{i0} = (x_{i0}, y_{i0})$, $i = 1, 2, \dots$

Let ρ and σ , $\rho > \sigma$, be given positive numbers.

Definition 1. A function $u(t) = (u_1(t), u_2(t), \dots)$, $t \in [0, T]$, with measurable coordinates $u_i(t) = (u_{i1}(t), u_{i2}(t))$, $i = 1, 2, \dots$, subject to the condition

$$\sum_{i=1}^{\infty} (u_{i1}^2(t) + u_{i2}^2(t)) \leq \rho^2, \quad 0 \leq t \leq T, \quad (5)$$

is referred to as the admissible control of pursuer.

Definition 2. A function $v(t) = (v_1(t), v_2(t), \dots)$, $t \in [0, T]$, with measurable coordinates $v_i(t) = (v_{i1}(t), v_{i2}(t))$, $i = 1, 2, \dots$, subject to the condition

$$\sum_{i=1}^{\infty} (v_{i1}^2(t) + v_{i2}^2(t)) \leq \sigma^2, \quad 0 \leq t \leq T, \quad (6)$$

is referred to as the admissible control of evader.

Let $\eta_i(t) = (x_i(t), y_i(t))$, $U_i = (U_{i1}, U_{i2})$, $v_i = (v_{i1}, v_{i2})$.

Definition 3. A strategy of pursuer is defined as a function of the form

$$U(t, v) = U^0(t) + v = (U_1^0(t) + v_1, U_2^0(t) + v_2, \dots), \quad (7)$$

where $U^0(t) = (U_1^0(t), U_2^0(t), \dots)$, $U_i^0(t) = (U_{i1}^0(t), U_{i2}^0(t))$, has measurable coordinates $U_i^0(t)$, $0 \leq t \leq T$, that satisfy the condition

$$\sum_{i=1}^{\infty} \left((U_{i1}^0(t))^2 + (U_{i2}^0(t))^2 \right) \leq (\rho - \sigma)^2, \quad 0 \leq t \leq T. \quad (8)$$

It is clear that

$$e^{A_i t} = e^{-\lambda_i t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}, \quad A_i = \begin{bmatrix} -\lambda_i & 1 \\ 0 & -\lambda_i \end{bmatrix}, \quad i = 1, 2, \dots \quad (9)$$

The function $\eta(t) = (\eta_1(t), \eta_2(t), \dots)$ defined by the equation

$$\eta_i(t) = e^{A_i t} \eta_{i0} + \int_0^t e^{A_i(t-s)} (u_i(s) - v_i(s)) ds, \quad i = 1, 2, \dots,$$

is the unique solution of (4) in $C(0, T; l_2)$, which is the class of continuous functions on $[0, T]$ that take their values in l_2 . The representation

$$\eta_i(t) = e^{A_i t} \left[\eta_{i0} + \int_0^t e^{-A_i s} (u_i(s) - v_i(s)) ds \right], \quad i = 1, 2, \dots \quad (10)$$

clearly, implies that $\eta(t) = 0$ if and only if $\gamma(t) = 0$, where

$$\gamma(t) = (\gamma_1(t), \gamma_2(t), \dots), \quad \gamma_i(t) = \eta_{i0} + \int_0^t e^{-A_i s} (u_i(s) - v_i(s)) ds, \quad (11)$$

3. RESULTS AND DISCUSSION

Let, for $i = 1, 2, \dots$,

$$\begin{aligned} W_i(\theta) &= \int_0^\theta e^{-A_i s} e^{-A_i^* s} ds = \int_0^\theta \begin{bmatrix} e^{\lambda_i s} & -e^{\lambda_i s} s \\ 0 & e^{\lambda_i s} \end{bmatrix} \begin{bmatrix} e^{\lambda_i s} & 0 \\ -e^{\lambda_i s} s & e^{\lambda_i s} \end{bmatrix} ds \\ &= \begin{bmatrix} \int_0^\theta e^{2\lambda_i s} (1 + s^2) ds & -\int_0^\theta e^{2\lambda_i s} s ds \\ -\int_0^\theta e^{2\lambda_i s} s ds & \int_0^\theta e^{2\lambda_i s} ds \end{bmatrix} = \begin{bmatrix} \psi_{11}(\theta) & \psi_{12}(\theta) \\ \psi_{21}(\theta) & \psi_{22}(\theta) \end{bmatrix}, \end{aligned}$$

where A^* is the transpose of the matrix A , and

$$\begin{aligned} \psi_{11}(\theta) &= \int_0^\theta e^{2\lambda_i s} (1 + s^2) ds, & \psi_{12}(\theta) &= -\int_0^\theta e^{2\lambda_i s} s ds, \\ \psi_{21}(\theta) &= -\int_0^\theta e^{2\lambda_i s} s ds, & \psi_{22}(\theta) &= \int_0^\theta e^{2\lambda_i s} ds. \end{aligned}$$

For the matrix $W_i(t)$ and $\eta_{i0} = (x_{i0}, y_{i0})$ the following relation holds

$$\eta_{i0} W_i(t) \eta_{i0} = \int_0^t |e^{-A_i^* s} \eta_{i0}|^2 ds.$$

It is obvious that if $\eta_{i0} \neq 0$, then this integral is positive. Hence, the matrix $W_i(t)$ is positive defined. Thereby, its determinant is positive

$$|W_i(t)| = \psi_{11}(t)\psi_{22}(t) - \psi_{12}^2(t) > 0. \quad (12)$$

Hence, the matrix $W_i(t)$ is invertible and it is not difficult to see that

$$W_i^{-1}(t) = \begin{bmatrix} \frac{\psi_{22}(t)}{|W_i(t)|} & -\frac{\psi_{12}(t)}{|W_i(t)|} \\ -\frac{\psi_{21}(t)}{|W_i(t)|} & \frac{\psi_{11}(t)}{|W_i(t)|} \end{bmatrix}.$$

In this section, we present the main result of our paper, which is the following statement about the guaranteed pursuit time.

Definition 5. We call number τ a guaranteed evasion time if, for any number τ' , $0 \leq \tau' < \tau$, one can construct an admissible control $v_0(t)$ of evader such that, for any admissible control of the pursuer, we have $\eta(t) \neq 0$ for all $0 \leq t \leq \tau'$ and $i = 1, 2, \dots$

The evader tries to minimize the guaranteed evasion time. The research questions are 1) is it possible to find a guaranteed evasion time τ in the game for any initial state $\eta_0 = (\eta_{10}, \eta_{20}, \dots) \neq 0$? 2) can we construct an admissible control for the evader that ensures the guaranteed evasion time?

Theorem 3.1. The number

$$\tau = \sup_{j=1,2,\dots} \tau_j,$$

where τ_j is a nonnegative root of the equation

$$\int_0^t e^{\lambda_j s} \left(\rho(1+s) - \frac{\sigma}{1+s} \right) ds = |\eta_{j0}|, \quad (13)$$

is a guaranteed evasion time in game (4)

Proof. First, we show that equation (13) has a unique nonnegative root. Clearly, if $\eta_{i0} = 0$, then equation (13) has the unique root $\tau_j = 0$.

Let $\eta_{j0} \neq 0$. Since $\rho > \sigma$ and $\lambda_j > 0$, we obtain

$$\begin{aligned} |\eta_{j0}| &= \int_0^{\tau_j} e^{\lambda_j s} \left(\rho(1+s) - \frac{\sigma}{1+s} \right) ds > \int_0^{\tau_j} e^{\lambda_j s} (\rho(1+s) - \sigma) ds > \\ &> \int_0^{\tau_j} e^{\lambda_j s} (\rho(1+s) - \sigma) ds > \int_0^{\tau_j} e^{\lambda_j s} \rho s ds > \rho \int_0^{\tau_j} s ds = \rho \frac{\tau_j^2}{2} \end{aligned} \quad (14)$$

Thus, the root of equation (13) satisfies the condition $\tau_j^2 < 2|\eta_{j0}|/\rho$. Consequently, $\tau = \sup_{j=1,2,\dots} \tau_j < \infty$.

Next, we construct a control for the evader that ensures $\eta(t) \neq 0$ on $[0, \tau']$ for the arbitrary control $u = u(t)$ of pursuer, where τ' is an arbitrary time that satisfy the condition $0 < \tau' < \tau$.

By definition of τ , we can find $j \in \{1, 2, \dots\}$ such that $\tau' < \tau_j$. We show that game (4) is never completed on $[0, \tau_j)$. Observe $\eta_{j0} \neq 0$ since otherwise $\tau_j = 0$ contradicting positivity of τ_j .

Let the the evader applies the following control

$$v_j(t) = -\sigma \frac{1}{1+t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} e_j, \quad v_i(t) = 0, \quad i = 1, 2, \dots, i \neq j, \quad t \in [0, \tau_j). \quad (15)$$

where $e_j = \frac{\eta_{j0}}{|\eta_{j0}|}$ is unit 2-vector, i.e., $e_j = (e_{j1}, e_{j2})$, $|e_j| = 1$.

Next, we establish that control (15) is the admissible. Using (6), we obtain

$$\begin{aligned} |v_j(t)| &= \left| -\sigma \frac{1}{1+t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} e_j \right| = \frac{\sigma}{1+t} \left\| \begin{bmatrix} e_{j1} + te_{j2} \\ e_{j2} \end{bmatrix} \right\| \\ &\leq \frac{\sigma}{1+t} \left(\left\| \begin{bmatrix} e_{j1} \\ e_{j2} \end{bmatrix} \right\| + \left\| \begin{bmatrix} te_{j2} \\ 0 \end{bmatrix} \right\| \right) = \frac{\sigma}{1+t} (1 + t|e_{j2}|) \\ &\leq \frac{\sigma}{1+t} (1+t) = \sigma \end{aligned}$$

Therefore, control (15) is admissible.

According to (10) and (11), $\eta(t) \neq 0$ if and only if $\gamma(t) \neq 0$. It suffices therefore to show that, for any admissible control of the pursuer $u(\cdot)$, $\gamma(t) \neq 0$, $t \in [0, \tau_j)$. Indeed,

$$\begin{aligned} \gamma_j(t) &= \eta_{j0} + \int_0^t e^{-A_j s} u_j(s) ds - \int_0^t e^{-A_j s} v_j(s) ds \\ &= \eta_{j0} + \int_0^t e^{-A_j s} u_j(s) ds + \sigma e_j \int_0^t \frac{1}{1+s} e^{-\lambda_j s} ds. \end{aligned} \quad (16)$$

Then, by (15) we have

$$\begin{aligned} |(e^{-A_j s} u_j(s), e_j)| &\leq |e^{-A_j s} u_j(s)| |e_j| = |(e^{-A_j s} u_j(s)| \\ &= e^{\lambda_j s} \left\| \begin{bmatrix} 1 & -s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_{j1} \\ u_{j2} \end{bmatrix} \right\| = e^{\lambda_j s} \left\| \begin{bmatrix} u_{j1} - s u_{j2} \\ u_{j2} \end{bmatrix} \right\| \\ &\leq e^{\lambda_j s} \left\| \begin{bmatrix} u_{j1} \\ u_{j2} \end{bmatrix} + \begin{bmatrix} -s u_{j2} \\ 0 \end{bmatrix} \right\| \leq \rho e^{\lambda_j s} (1+s) \end{aligned}$$

Consequently, using (16) and the fact that τ_j is the root of equation (13), we obtain for $t \in [0, \tau_j)$ that

$$\begin{aligned} (\gamma_j(t), e_j) &= |\eta_{j0}| + \int_0^t (e^{-A_j s} u_j(s), e_j) ds + \sigma \int_0^t \frac{1}{1+s} e^{\lambda_j s} ds \\ &\geq |\eta_{j0}| - \int_0^t \rho (1+s) e^{\lambda_j s} ds + \int_0^t \frac{\sigma}{1+s} e^{\lambda_j s} ds \\ &> |\eta_{j0}| - \int_0^{\tau_j} e^{\lambda_j s} \left(\rho(1+s) - \frac{\sigma}{1+s} \right) ds = 0. \end{aligned}$$

As the result, $\gamma_j(t) \neq 0$, $t \in [0, \tau_j)$, and hence $\eta_j(t) \neq 0$ by (11), therefore, $\eta(t) \neq 0$ for $t \in [0, \tau_j)$. In particular, $\eta(t) \neq 0$ on the interval $[0, \tau']$. The proof of

Theorem 3. 1 is complete.

□

4. CONCLUSION

This paper investigated an evasion-type differential game modeled by an infinite system of binary differential equations (4)

$$\dot{z}_i = C_i z_i + u_i - v_i, \quad i = 1, 2, \dots,$$

which correspond to the Jordan matrices $C_i = \begin{bmatrix} -\lambda_i & 1 \\ 0 & -\lambda_i \end{bmatrix}$, where λ_i , $i = 1, 2, \dots$, are real numbers. While previous studies have addressed similar problems for systems associated with certain matrix forms, this study extends the scope to a broader class of infinite-dimensional systems. A closed-form expression for the guaranteed evasion time has been derived.

Previous researches study only differential games described by infinite systems

$$\dot{z}_i = A_i z_i + u_i - v_i, \quad A_i = \begin{bmatrix} -\lambda_{i1} & 0 \\ 0 & -\lambda_{i2} \end{bmatrix}$$

$$\text{And} \quad \dot{z}_i = B_i z_i + u_i - v_i, \quad B_i = \begin{bmatrix} -\alpha_i & -\beta_i \\ \beta_i & -\alpha_i \end{bmatrix}.$$

It is well known that any 2x2-matrix can be reduced to one of the Jordan matrices A_i , B_i , and C_i . Therefore, the infinite system we have studied in the present paper fills the gap between the infinite 2-systems corresponding to the 2x2 Jordan matrices. Evasion differential game problems for the infinite system of differential equations corresponding to the 2x2 matrices C_i have been considered in the present paper for the first time.

We have studied an evasion differential game and obtained a formula for the guaranteed evasion time.

Future investigations may consider more general formulations involving second-order systems of the form:

$$\dot{z}_i = D_i z_i + u_i - v_i, \quad D_i = \begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix}, \quad i = 1, 2, \dots,$$

where a_i , b_i , c_i , and d_i are any given real numbers.

REFERENCES

- [1]. Albeverio S., Alimov Sh.A.(2008) On a time-optimal control problem associated with the heat exchange process. *Applied mathematics and optimization*. 57(1): 58–68.
- [2]. Avdonin S.A., Ivanov S.A. (1995) Families of Exponentials: The Method of Moments in Controllability Problems for Distributed Parameter Systems. Cambridge: Cambridge University Press.
- [3]. Butkovskiy A.G. (1969) Theory of Optimal Control of Distributed Parameter Systems. New York: Elsevier.
- [4]. Chaves-Silva F., Zhang X., Zuazua E. (2017) Controllability of evolution equations with memory. *SIAM J. on Control and Optimization*. 55(4): 2437–2459. DOI: 10.1137/151004239
- [5]. Chernous'ko F.L. (1992) Bounded controls in distributed-parameter systems, *Journal of Applied Mathematics and Mechanics*, 56(5), 707–723.
- [6]. Fattorini H.O. (1964) Time-Optimal control of solutions of operational differential equations. *SIAM J. Control*, 2(1): 54–59.
- [7]. Geshkovski B., Zuazua E. (2022). Turnpike in optimal control of PDEs, ResNets, and beyond. *Acta Numerica*. 31: 135-263. doi:10.1017/S0962492922000046
- [8]. Ibragimov G.I., Allahabi F., Kuchkarov A.Sh. (2014) A pursuit problem in an infinite system of second-order differential equations. *Ukrainian Mathematical Journal*. 65(8): 1203–1216.
- [9]. Lions J.L. (1968). *Contrôle Optimal de Systèmes Gouvernés par des Equations aux Dérivées Partielles*. Paris: Dunod.
- [10]. Liu W., Zhao T., Ito K., Zhang Zh. (2023). Error estimates of Fourier finite volume element method for parabolic Dirichlet boundary optimal control problems on complex connected domains. *Applied Numerical Mathematics*. 186: 164-201, doi.org/10.1016/j.apnum.2023.01.007.
- [11]. Osipov Yu.S. (1975) The theory of differential games in systems with distributed parameters. *Dokl Akad Nauk SSSR* 223(6): 1314–1317 [*SovietMath. Dokl.* 16, 1093–1097 (1975)].
- [12]. Philippe Martin, Lionel Rosier, Pierre Rouchon.(2013) Null controllability of the structurally damped wave equation with moving control. *SIAM J. Control Optim.* 51(1): 660–684. DOI: 10.1137/110856150.
- [13]. Salimi M., Ferrara M. (2019). Differential game of optimal pursuit of one evader by many pursuers. *International Journal of Game Theory*, 48(2), 481–490.
- [14]. Satimov, N.Yu., Tukhtasinov, M. (2006). Game problems on a fixed interval in controlled first-order evolution equations, *Mathematical notes*, 80(3-4), 578–589.
- [15]. Satimov, N.Yu., Tukhtasinov, M. (2005). On Some Game Problems for First-Order Controlled Evolution Equations, *Differential Equations*, 41(8), 1169–1177.
- [16]. Tukhtasinov M., Ibragimov G.I., Kuchkarova S.A., Hasim R.M. (2021). Differential games for an infinite 2-systems of differential equations. *Mathematics, Special Issue "Mathematical Game Theory 2021"*, 9(13), 1467

Vektorlar yordamida geometrik masalalar yechish

M.K.Maxkamov, J.D.Dexkonov, A.X.Akbarov

Andijon davlat universiteti

Annotatsiya. Bu ishda, vektorlar yordamida geometrik masalalarni yechish va ba'zi muhim tengsizliklarni isbotlash mumkinligi haqida ma'lumotlar keltirilgan. Unda vektorlarning asosiy xossalari, ular orasidagi burchak, skalyar va vektor ko'paytma tushunchalari orqali murakkab geometrik shakllar va ularning o'zaro bog'liqligi tahlil qilinadi.

Kalit so'zlar: Uchburchak, uchburchak medianasi, uchburchak bissektrisasi, aylana, Styuart teoremasi.

Abstract. In this work, information is provided on how to solve geometric problems and prove some important inequalities using basic vectors. It analyzes the mutual relationships of complex geometric figures through the fundamental properties of vectors, the angle between them, and the concepts of scalar and vector products.

Keywords: Triangle, median of a triangle, angle bisector of a triangle, circle, Stewart's Theorem.

Аннотация. В данной работе представлена информация о том, как решать геометрические задачи и доказывать некоторые важные неравенства с использованием базовых векторов. Анализируются взаимосвязи сложных геометрических фигур на основе фундаментальных свойств векторов, углов между ними, а также понятий скалярного и векторного произведения.

Ключевые слова: треугольник, медиана треугольника, биссектриса треугольника, окружность, теорема Стюарта.

I. INTRODUCTION

Geometrik masalalarni yechish, ya'ni uzunlik, burchak, masofa va yuzalar kabi kattaliklarni aniqlash muhim ahamiyatga ega. An'anaviy usullar bilan bu kabi masalalarni yechish ba'zan murakkab va ko'p bosqichli bo'lishi mumkin. Shu sababli zamonaviy matematik yondashuvlardan biri vektorlar yordamida yechim topish usuli keng qo'llanilmoqda.

Vektorlar fazodagi yo'nalgan kesma sifatida ta'riflanib, ularning algebraik xossalari geometriyadagi ko'plab muammolarni soddalashtirish imkonini beradi. Vektorlarning uzunligi, yo'nalishi, ularning skalyar va vektor ko'paytmalari orqali geometrik shakllar orasidagi metrik bog'liqliklar, burchaklar va masofalarni aniqlash mumkin. Ayniqsa, uchburchak, parallelogram, doira kabi shakllarga oid masalalarda vektor yondashuvi juda qulay va samarali hisoblanadi.

Ushbu maqolada vektorlar yordamida geometrik masalalarni yechish usullari, ularning nazariy asoslari, shuningdek ba'zi muhim geometrik tengsizliklarning vektorli isbotlari ko'rib chiqiladi. Maqsad o'quvchi va tadqiqotchilarga vektorlarning amaliy imkoniyatlarini ko'rsatish va geometriyani chuqurroq anglashga xizmat qiluvchi muqobil yondashuvlarni taqdim etishdir.

II. MASALANING QO'YISHI VA ASOSIY NATIJA

Uchburchak medianasi uzunligini hisoblash. Tomonlari

$$BC = a, CA = b, AB = c$$

bo'lgan ABC uchburchakda $AA_1 = m_a$ mediana o'tkazilgan bo'lsin. U holda

$$BA_1 = A_1C$$

bo'lgani uchun A nuqtani qutb deb qabul qilsak, vektorlarni ayirish qoidasiga asosan

$$\vec{A_1} - \vec{B} = \vec{C} - \vec{A_1}$$

vektor tenglikni yozamiz. Bundan

$$\vec{A_1} = \frac{\vec{B} + \vec{C}}{2}$$

bu tenglikni skalyar kvadratga ko'tarib

$$\vec{A_1}^2 = \frac{\vec{B}^2 + 2\vec{B} \cdot \vec{C} + \vec{C}^2}{4}$$

tenglikni yozamiz. Ammo

$$2\vec{B} \cdot \vec{C} = 2cb \cos \angle A = 2bc \cdot \frac{b^2 + c^2 - a^2}{2bc} =$$

$$= b^2 + c^2 - a^2, \vec{B}^2 = c^2, \vec{C}^2 = b^2, \vec{A_1}^2 = m_a^2$$

ekanligini e'tiborga olsak, bu tenglikdan

$$m_a^2 = \frac{2(b^2 + c^2) - a^2}{4}$$

va bundan $AA_1 = m_a$ mediananing uzunligi

$$m_a = \frac{1}{2} \sqrt{2(b^2 + c^2) - a^2}$$

formula bo'yicha hisoblanishi mumkinligini ko'ramiz.

Uchburchak bissektrisasi uzunligini hisoblash. Tomonlari

$$BC = a, CA = b$$

va $AB = c$ bo'lgan ABC uchburchakda $AA_1 = \beta_a$ bissektrisa o'tkazilgan bo'lsin. U holda bissektrisa xossasiga ko'ra

$$BA_1 : BA = A_1C : CA,$$

yoki A nuqtani qutb deb qabul qilib, vektorlarni ayirish qoidasiga asosan

$$b(\vec{A_1} - \vec{B}) = c(\vec{C} - \vec{A_1})$$

vektor tenglikni yozamiz. Bundan

$$\vec{A_1} = \frac{b\vec{B} + c\vec{C}}{b+c}$$

Bu tenglikni skalyar kvadratga ko'tarib

$$\vec{A_1}^2 = \frac{b^2\vec{B}^2 + 2bc\vec{B} \cdot \vec{C} + c^2\vec{C}^2}{(b+c)^2}$$

tenglikni yozamiz. Ammo

$$2\vec{B} \cdot \vec{C} = 2cb \cos \angle A = 2bc \cdot \frac{b^2 + c^2 - a^2}{2bc} =$$

$$= b^2 + c^2 - a^2, \vec{B}^2 = c^2, \vec{C}^2 = b^2, \vec{A_1}^2 = \beta_a^2$$

ekanligini e'tiborga olsak, bu tenglikdan

$$\begin{aligned} \beta_a^2 &= \frac{2b^2c^2 + bc(b^2 + c^2 - a^2)}{(b+c)^2} = \\ &= \frac{bc[(b+c)^2 - a^2]}{(b+c)^2} = \\ &= \frac{bc(b+c+a)(b+c-a)}{(b+c)^2} = \frac{4bcp(p-a)}{(b+c)^2}. \end{aligned}$$

Demak, $AA_1 = \beta_a$ bissektrisaning uzunligi

$$\beta_a = \frac{2}{b+c} \sqrt{bcp(p-a)}$$

formula bo'yicha hisoblanishi mumkinligini ko'ramiz, bu yerda

$$p = \frac{a+b+c}{2}$$

uchburchakning yarim perimetri.

Uchburchakka tashqi va ichki chizilgan aylanalar markazlari orasidagi masofani hisoblash. Bu masofa geometriyada Eyler teoremasi deb nomlangan quyidagi teoremda mujassamlashgan:

Teorema. Uchburchakka tashqi chizilgan aylana markazi bilan ichki chizilgan aylana markazi orasidagi masofa uchun

$$d = \sqrt{R^2 - 2Rr} \quad (1)$$

formula o'rinli, bu yerda R va r mos ravishda uchburchakka tashqi va ichki chizilgan aylana radiuslari.

Isboti.

$$BC = a, CA = b, AB = c$$

va AA_1 va CC_1 $\triangle ABC$ ning bissektrisalari bo'lsin.

Uchburchak bissektrisasi xossasiga ko'ra

$$AC_1 : C_1B = b : a$$

va

$$BA_1 : A_1C = c : b$$

$$AC_1 = \frac{bc}{a+b}, AC = b$$

ekanligidan

$$\frac{CI}{IC_1} = \frac{CA}{AC_1} = \frac{b}{\frac{bc}{a+b}} = \frac{a+b}{c}$$

Teoremani isbotlashni davom ettirish uchun vektorlar usulidan foydalanamiz.

Dastlab uchburchak bissektrisa xossasiga ko'ra

$$\frac{\vec{AC_1}}{b} = \frac{\vec{C_1B}}{a} \text{ va } \frac{\vec{CI}}{\vec{IC_1}} = \frac{a+b}{c}$$

vector tengliklarni yozamiz. Bu tengliklardan vektorlarni ayirish qoidasi bo'yicha uchburchakka tashqi chizilgan aylana markazi O nuqtani qutb deb hisoblab,

$$\vec{C_1} = \frac{a}{a+b} \vec{A} + \frac{b}{a+b} \vec{B}$$

va

$$(a+b+c)\vec{I} = (a+b)\vec{C_1} + c\vec{C}$$

bu tengliklardan

$$(a+b+c)\vec{I} = a\vec{A} + b\vec{B} + c\vec{C}.$$

Oxirgi tenglikni skalyar kvadratga ko'tarib

$$\begin{aligned} (a+b+c)^2 OI^2 &= (a^2 + b^2 + c^2) R^2 + \\ &+ 2ab\vec{A} \cdot \vec{B} + 2bc\vec{B} \cdot \vec{C} + 2ca\vec{C} \cdot \vec{A} \end{aligned}$$

tenglikni hosil qilamiz.

Ammo skalyar ko'payitma ta'rifi va kosinuslar teoremasiga ko'ra

$$2\vec{A} \cdot \vec{B} = 2R \cdot R \cdot \cos \angle AOB =$$

$$= 2R^2 \cdot \frac{R^2 + R^2 - c^2}{2R \cdot R} = 2R^2 - c^2$$

Xuddi shu kabi

$$2\vec{B} \cdot \vec{C} = 2R^2 - a^2, \quad 2\vec{C} \cdot \vec{A} = 2R^2 - b^2$$

bo'lgani uchun

$$\begin{aligned} (a+b+c)^2 OI^2 &= \\ &= (a+b+c)^2 R^2 - abc(a+b+c) \end{aligned}$$

bundan

$$\begin{aligned} OI^2 &= R^2 - \frac{abc}{a+b+c} = \\ &= R^2 - \frac{abc}{4S} \cdot \frac{4S}{2p} = R^2 - 2Rr \end{aligned}$$

Oxirgi tenglikdan uchburchakka tashqi chizilgan aylana markazi bilan ichki chizilgan aylana markazi orasidagi masofa uchun (1) formula o'rinli ekanligiga ishonch hosil qilamiz.

Styuart teoremasini isbotlash.

Uchburchakdagi ko'plab metrik munosabatlarni topishda uchburchak uchidan qarshisidagi tomonda yotgan nuqtagacha bo'lgan masofani hisoblash talab qilinadi. Bunday masalalarni yechishning usullaridan biri Styuart teoremasidir:

Teorema. Agar ABC uchburchakda

$$BC = a, CA = b, AB = c$$

va $D \in BC$ bo'lsa, u holda

$$AD^2 \cdot a = c^2 \cdot DC + b^2 \cdot BD - BD \cdot DC \cdot a$$

munosabat o'rinli.

Isboti. A nuqta qutb bo'lsin. Agar

$$BD : DC = m : n$$

bo'lsa, u holda $BD + DC = a$ bo'lgani uchun

$$BD = \frac{am}{m+n}, \quad DC = \frac{an}{m+n}$$

U holda

$$\frac{\vec{BD}}{m} = \frac{\vec{DC}}{n}$$

tenglikdan vektorlarni ayirish qoidasiga ko'ra

$$n(\vec{D} - \vec{B}) = m(\vec{C} - \vec{D}) \Rightarrow$$

$$\vec{D} = \frac{n}{m+n} \vec{B} + \frac{m}{m+n} \vec{C} \Rightarrow$$

$$\vec{D} = \frac{DC}{a} \vec{B} + \frac{BD}{a} \vec{C}$$

Oxirgi tenglikni kvadratga ko'tarib,

$$\begin{aligned} \vec{D}^2 &= \frac{DC^2}{a^2} \vec{B}^2 + \frac{BD^2}{a^2} \vec{C}^2 + 2 \frac{BD \cdot DC}{a^2} \vec{B} \cdot \vec{C} = \\ &= \frac{DC^2}{a^2} c^2 + \frac{BD^2}{a^2} b^2 + \frac{BD \cdot DC}{a^2} (b^2 + c^2 - a^2) \end{aligned}$$

yoki

$$\begin{aligned} AD^2 &= \frac{DC^2 + BD \cdot DC}{a^2} c^2 + \\ &+ \frac{BD^2 + BD \cdot DC}{a^2} b^2 - BD \cdot DC \end{aligned}$$

Bundan Styuart teoremasiga teng kuchli bo'lgan

$$AD^2 = \frac{DC}{a} c^2 + \frac{BD}{a} b^2 - BD \cdot DC$$

munosabatni hosil qilamiz. Teorema isbotlandi.

Vektorlarni qo'llab turli mazmundagi geometrik masalalar yechish uchun o'quvchilarga quyidagi adabiyotlarni tavsiya etamiz.

REFERENCES

- [1] Клопский В.М., Ягодовский М.И., Скопец З.А., Об изучении Векторы в IX классе // Математика в школе, 1981, №1, с.22-28.
- [2] Клопский В.М., Ягодовский М.И., Скопец З.А., Об изучении *Векторы в IX классе // Математика в школе, 1981, №2, с.35-41.
- [3] Скопец З.А., Теорема косинусов для трехгранного угла // Математика в школе, 1970, №6, с.69-70.
- [4] Кушнir И.А., О применении одной векторной формулы // Математика в школе, 1981, №2, с.32-34.
- [5] Крупин П.А., Задачи на принадлежность точек прямой и плоскости // Математика в школе, 1982, №6, с.51-54.
- [6] Готман Э.Г., Задача о правильном шестиугольнике и ее обобщение // Математика в школе, 1982, №6, с.54-56.
- [7] Имраев А.Г., Применении векторов в решению задач на вычисление расстояний и углов. // Математика в школе, 1984, №2, с.64-67.
- [8] Ясиновой Э.А., Применении векторов. // Математика в школе, 1979, №3, с.71-73.
- [9] Ольхов В.Е., Об использование тригонометрических функций при доказательство неравенств // Математика в школе, 1979, №2, с.56.
- [10] Майоров В.М., Скопец З.А. Векторное решение геометрических задач, Просвещение 1968.

О разрешимости краевых задач для вязко-трансзвукового уравнения с несимметричными условиями по времени

Иброхимов Хусниддин Комилжон угли

Наманганский государственный технический университет.

E-mail: xusniddin571@mail.ru

Аннотация. В данной работе исследованы краевые задачи для вязко-трансзвукового уравнения с несимметричными условиями по времени. Единственность решения поставленных задач доказаны с помощью методом интегралов энергии. Показаны, что при нарушении условий единственности существуют нетривиальные решения однородных задач. Существование решения установлены методом разделения переменных. Решение задачи представлены в виде бесконечного ряда с обоснованием возможности его почленного дифференцирования по всем переменным. При доказательстве равномерной сходимости установлены отличность от нуля "малого знаменателя".

Ключевые слова: Уравнение с кратными характеристиками, краевая задача, единственность, существование, метод разделения переменных, собственное значение, собственная функция, функциональный ряд, абсолютная и равномерная сходимость.

Annotatsiya: Ushbu ishda vaqt bo'yicha simmetrik bo'lmagan shartlar bilan qovushqoq-trans tovushli tenglama uchun chegaraviy masalalar o'rganilgan. Energiya integrallari usuli yordamida yechimning yagona ekanligi isbotlangan. Yagonalik shartlari buzilganda bir jinsli masala notrivial yechimga ega bo'lishi ko'rsatilgan. O'zgaruvchilarni ajratish usulidan foydalanib, yechimning mavjudligi isbotlangan. Masalaning yechimi cheksiz qator ko'rinishida ifodalangan bo'lib, uning barcha o'zgaruvchilar bo'yicha hadma-had differensiallash mumkinligi asoslangan. Tekis yaqinlashuvchi ekanligini isbotlash jarayonida "kichik maxraj"ning noldan farqli ekanligi aniqlangan.

Kalit so'zlar: Karrali xarakteristik tenglama, chegaraviy masala, yagonalik, mavjudlik, o'zgaruvchilarni ajratish usuli, xos son, xos funksiya, funktsional qator, absolyut va tekis yaqinlashish.

Annotation. This work examines boundary value problems for the viscous-transonic equation with non-symmetric temporal constraints. Uniqueness of solutions is rigorously proved via the energy integral method. We establish that when uniqueness conditions are violated, non-trivial solutions emerge for the associated homogeneous problems. Existence of solutions is confirmed through the method of separation of variables. The solution is represented as a convergent infinite series, with complete justification provided for its term-by-term differentiability across all independent variables. The uniform convergence analysis identifies the persistent non-degeneracy of small denominator terms.

Key words: Equations with multiple characteristics, boundary value problem, uniqueness, existence, method of separated variables, eigenvalue, eigenfunction, functional series, absolute and uniform convergence.

I. Введение

Дифференциальные уравнения в частных производных третьего порядка рассматриваются при решении задач теории нелинейной акустики и в гидродинамической теории космической плазмы, фильтрации жидкости в пористых средах [1].

Довольно часто резкие изменения параметров потока происходят в узких областях, прилегающих к ударным волнам. Градиенты параметров потока в них могут быть настолько значительными, что наряду с нелинейным характером движения становится необходимым учитывать влияние вязкости и теплопроводности. Такие течения называются короткими волнами. К теории коротких волн относится теория трансзвуковых течений.

Следует отметить, что в последнее время в литературе это уравнение все больше называют вязким трансзвуковым уравнением, или просто ВТ-уравнением [2]. Общая теория коротких волн и ее связь с трансзвуковыми течениями дана в [3]. В ряде конкретных физических задач, рассмотрены примеры коротких волн, возникающих в стационарных течениях, где происходят диссипативные процессы. В [4] рассматривается маховское отражение слабых ударных волн от клина. В [5-6] рассмотрено взаимодействие слабых ударных волн в неограниченном слое.

В работе [7], учитывая свойства вязкости и теплопроводности газа, из системы Навье-Стокса было получено уравнение третьего порядка с кратными

характеристиками, содержащее вторую производную по времени

$$u_{xxx} + u_{yy} - \frac{\nu}{y} u_y = u_x u_{xx}, \quad \nu = \text{const.}$$

Это уравнение при $\nu = 1$ описывает осесимметричный поток, а при $\nu = 0$ описывает плоско - параллельный поток [8].

Первые результаты по уравнению третьего порядка с кратными характеристиками были получены в работах Н. Block [9], E. Del Vecchio [10].

L. Catabriga в работе [11] для уравнения $D_x^{2n+1}u - D_y^2u = 0$ построил фундаментальное решение в виде двойного несобственного интеграла и изучил свойства потенциала, решил краевые задачи.

В работах [12-13] построены фундаментальные решения для уравнения третьего порядка с кратными

$$u_y(x, 0) = u(x, q) = 0, \quad 0 \leq x \leq p, \quad (2)$$

$$\begin{cases} au(0, y) + bu_{xx}(0, y) = \psi_1(y), \\ cu(p, y) + du_{xx}(p, y) = \psi_2(y), \quad 0 \leq y \leq q, \\ u_x(0, y) = \psi_3(y), \end{cases} \quad (3)$$

где a, b, c и d - заданные постоянные, а также $a^2 + b^2 \neq 0, c^2 + d^2 \neq 0$, а $\psi_i(y), i = \overline{1, 3}$, - заданные достаточно гладкие функции, причем

$$\psi'_{in}(0) = \psi_{in}(q) = 0, \quad i = \overline{1, 3}. \quad (4)$$

Отметим, что аналогичная задача для сопряженного уравнения исследованы в работах [14-16]. Краевые задачи, близкие к теме данной работы, изучались в [17]. В работе [18] рассмотрен случай $a = 1, b = 0, c = 1, d = 0$, а в работе [19] случай $a = 0, b = 1, c = 1, d = 0$.

III. Единственность решения

Теорема 1. Если задача A_1 имеет решение, то при выполнении условий $ab \leq 0, cd \geq 0, \mu \leq 0$, оно единственно.

Доказательство. Предположим обратное, пусть задача A_1 имеет два решения $u_1(x, y)$ и $u_2(x, y)$. Тогда функция $u(x, y) = u_1(x, y) - u_2(x, y)$ удовлетворяет уравнению (1) с однородными краевыми условиями. Докажем, что $u(x, y) \equiv 0$ в $\overline{D}$.

В области D справедливо тождество

$$u L[u] \equiv \frac{\partial}{\partial x} \left(u u_{xx} - \frac{1}{2} u_x^2 \right) + \frac{\partial}{\partial y} (u u_y) - u_y^2 + \mu u^2 = 0. \quad (5)$$

Интегрируя тождество (5) по области D , имеем

характеристиками, содержащие вторые производные по времени, выраженные через вырожденные гипергеометрические функции, изучены их свойства, найдены оценки при $|t| \rightarrow \infty$.

II. Постановка задачи

В области $D = \{(x, y): 0 < x < p, 0 < y < q\}$

рассмотрим уравнение

$$L[u] \equiv u_{xxx} + u_{yy} + \mu u = 0, \quad (1)$$

где $p, q \in R$ и для него исследуем следующую задачу.

Задача A_1 . Найти функцию $u(x, y)$ из класса $C_{x,y}^{3,2}(D) \cap C_{x,y}^{2,1}(\overline{D})$, удовлетворяющую уравнению (1) в области D и следующим краевым условиям:

$$\int_0^p \int_0^q \frac{\partial}{\partial x} \left[u(x, y) u_{xx}(x, y) - \frac{1}{2} u_x^2(x, y) \right] dx dy +$$

$$+ \int_0^p \int_0^q \frac{\partial}{\partial y} \left[u(x, y) u_y(x, y) \right] dx dy - \int_0^p \int_0^q u_y^2(x, y) dx dy + \mu \int_0^p \int_0^q u(x, y) dx dy = 0.$$

Учитывая однородные краевые условия и требуя $a \neq 0$, $c \neq 0$, получим

$$\frac{d}{c} \int_0^q u_{xx}^2(p, y) dy - \frac{b}{a} \int_0^q u_{xx}^2(0, y) dy + \frac{1}{2} \int_0^q u_x^2(p, y) dy + \iint_D u_y^2 dx dy - \mu \iint_D u^2 dx dy = 0.$$

Пусть $\mu = 0$, тогда по условиям теоремы 1, т.е. $ab \leq 0$, $cd \geq 0$, получим $u_y(x, y) = 0$, отсюда имеем $u(x, y) = f(x)$. Пологая $y = 0$ и учитывая что $u(x, 0) = 0$, имеем $f(x) = 0$. Следовательно, $u(x, y) \equiv 0$, $(x, y) \in \bar{D}$. Если $\mu < 0$, то из пятого слагаемого, получим $u(x, y) \equiv 0$, $(x, y) \in \bar{D}$. В силу последнего, получим $u_1(x, y) = u_2(x, y)$.

Замечание 1. Отметим, что при нарушении условий теоремы 1, т.е. $a = bk_n^2$, $b > 0$, $k_n > 0$, $c, d \in R$, $c^2 + d^2 \neq 0$ однородная задача A_1 для однородного уравнения (1) может иметь нетривиальное решение. Например, уравнения (1) с однородными краевыми условиями:

$$u_y(x, 0) = u(x, q) = 0, \quad (2)$$

$$\begin{cases} k_n^2 u(0, y) + u_{xx}(0, y) = 0, \\ cu(p, y) + du_{xx}(p, y) = 0, \\ u_x(0, y) = 0, \end{cases} \quad (3)$$

имеет нетривиальное решение в виде

$$u(x, y) = 2e^{-\frac{k_n}{2}x} \cos\left(\frac{\sqrt{3}}{2}k_n x - \frac{\pi}{6}\right) \cos\frac{(2n-1)\pi}{2q}y, \quad n \in N.$$

В случаях $b \neq 0$, $d \neq 0$; $a \neq 0$, $d \neq 0$; $c \neq 0$, $b \neq 0$, аналогично получаем равенство $u(x, y) \equiv 0$ в $\bar{D}$.

Теорема 1 доказана.

IV. Существование решения

Теорема 2. Если функции $\psi_i(y) \in C^2(0 \leq y \leq q)$, $i = \overline{1, 3}$ и выполняется условия (4), то решение задачи A_1 существует.

Доказательство. Решение задачи A_1 ищем в виде

$$u(x, y) = X(x)Y(y). \quad (6)$$

Подставляя (6) в уравнение (1) и разделяя переменные, относительно функции $Y(y)$ получим задачу типа Штурма-Лиувилля:

$$\begin{cases} Y'' + \lambda Y = 0, \\ Y'(0) = Y(q) = 0, \end{cases} \quad (7)$$

где λ - параметр разделения.

Нетривиальные решения задачи (7), существует только при

$$\lambda_n = \left(\frac{(2n-1)\pi}{2q} \right)^2, \quad n \in N.$$

Эти числа являются собственными значениями задачи (7), а соответствующие им собственные функции имеют вид [20]

$$Y_n(y) = \sqrt{\frac{2}{q}} \cos\left(\frac{(2n-1)\pi}{2q} y\right). \quad (8)$$

Отметим, что система собственных функций (8) задачи (7) является полной и ортонормированной в пространстве $L_2(0, q)$ [21].

Для $X_n(x)$ приходим к следующей задаче:

$$\begin{cases} X_n''' - (\lambda_n - \mu)X_n = 0, \\ aX_n(0) + bX_n''(0) = \psi_{1n}, \\ cX_n(p) + dX_n''(p) = \psi_{2n}, \\ X_n'(0) = \psi_{3n}, \end{cases} \quad (9)$$

где

$$\psi_{in} = \sqrt{\frac{2}{q}} \int_0^q \psi_i(y) \cos\left(\frac{(2n-1)\pi}{2q} y\right) dy, \quad i = \overline{1,3}, \quad n \in N.$$

Общее решение уравнение в задачи (9), имеет вид

$$X_n(x) = C_{1n} e^{k_n x} + e^{-\frac{1}{2}k_n x} \left(C_{2n} \cos \frac{\sqrt{3}}{2} k_n x + C_{3n} \sin \frac{\sqrt{3}}{2} k_n x \right), \quad (10)$$

здесь

$$k_n = \sqrt[3]{\lambda_n - \mu} = \sqrt[3]{\left(\frac{(2n-1)\pi}{2q}\right)^2 - \mu}, \quad k_n > 0, \quad k_n = O\left(n^{\frac{2}{3}}\right), \quad n \in N.$$

Учитывая условия задачи (9), получим:

$$\begin{cases} C_{1n} \left(a + bk_n^2 \right) + C_{2n} \left(a - \frac{bk_n^2}{2} \right) - C_{3n} \frac{\sqrt{3}bk_n^2}{2} = \psi_{1n}, \\ C_{1n} e^{k_n p} \left(c + dk_n^2 \right) + C_{2n} e^{-\frac{1}{2}k_n p} \left(c \cos\left(\frac{\sqrt{3}}{2} k_n p\right) + dk_n^2 \cos\left(\frac{\sqrt{3}}{2} k_n p - \frac{2\pi}{3}\right) \right) + \\ + C_{3n} e^{-\frac{1}{2}k_n p} \left(c \sin\left(\frac{\sqrt{3}}{2} k_n p\right) + dk_n^2 \sin\left(\frac{\sqrt{3}}{2} k_n p - \frac{2\pi}{3}\right) \right) = \psi_{2n}, \\ k_n C_{1n} - \frac{1}{2} k_n C_{2n} + \frac{\sqrt{3}}{2} k_n C_{3n} = \psi_{3n}. \end{cases} \quad (11)$$

Итак, для определения коэффициентов $C_{in}, i = \overline{1,3}$, мы получили систему алгебраических уравнений (11).

Введем обозначение:

$$\alpha_n = \cos\left(\frac{\sqrt{3}}{2}k_n p\right), \quad \beta_n = \sin\left(\frac{\sqrt{3}}{2}k_n p\right), \quad \gamma_n = \cos\left(\frac{\sqrt{3}}{2}k_n p - \frac{2\pi}{3}\right),$$

$$\delta_n = \sin\left(\frac{\sqrt{3}}{2}k_n p - \frac{2\pi}{3}\right).$$

Тогда (11) имеет следующий вид:

$$\begin{cases} C_{1n} \left(a + bk_n^2\right) + C_{2n} \left(a - \frac{bk_n^2}{2}\right) - C_{3n} \frac{\sqrt{3}bk_n^2}{2} = \psi_{1n}, \\ C_{1n}e^{k_n p} \left(c + dk_n^2\right) + C_{2n}e^{-\frac{1}{2}k_n p} \left(c\alpha_n + dk_n^2\gamma_n\right) + C_{3n}e^{-\frac{1}{2}k_n p} \left(c\beta_n + dk_n^2\delta_n\right) = \psi_{2n}, \\ k_n C_{1n} - \frac{1}{2}k_n C_{2n} + C_{3n} \frac{\sqrt{3}}{2}k_n = \psi_{3n}. \end{cases} \quad (12)$$

Основной определитель системы (12) имеет

$$\Delta(n) = \begin{vmatrix} a + bk_n^2 & a - \frac{1}{2}bk_n^2 & -\frac{\sqrt{3}}{2}bk_n^2 \\ e^{k_n p} (c + dk_n^2) & e^{-\frac{k_n}{2}p} (c\alpha_n + dk_n^2\gamma_n) & e^{-\frac{k_n}{2}p} (c\beta_n + dk_n^2\delta_n) \\ k_n & -\frac{1}{2}k_n & \frac{\sqrt{3}}{2}k_n \end{vmatrix} = \frac{\sqrt{3}}{2}k_n^5 e^{k_n p} \bar{\Delta}(n),$$

где

$$\bar{\Delta}(n) = \left(\frac{c}{k_n^2} + d\right) \left(b - \frac{a}{k_n^2}\right) + e^{-\frac{3k_n}{2}p} \left\{ \left(\frac{ac}{k_n^4} - \frac{2ad}{k_n^2} + \frac{2bc}{k_n^2} - bd\right) \cos \frac{\sqrt{3}}{2}k_n p + \sqrt{3} \left(\frac{ac}{k_n^4} + bd\right) \sin \frac{\sqrt{3}}{2}k_n p \right\},$$

Покажем, что $\Delta \neq 0$. Для этого докажем следующую лемму:

Лемма. Краевая задача

$$\begin{cases} X_n''' - (\lambda_n - \mu)X_n = 0, \\ aX_n(0) + bX_n''(0) = 0, \\ cX_n(p) + dX_n''(p) = 0, \\ X_n'(0) = 0, \end{cases} \quad (13)$$

имеет только тривиальное решение.

Доказательство лемма. Предположим обратное, пусть $X_n(x) \neq 0$.

Рассмотрим тождество

$$X_n(X_n''' - (\lambda_n - \mu)X_n) = 0$$

или

$$\left(X_n X_n'' - \frac{1}{2} (X_n')^2 \right)' - (\lambda_n - \mu) X_n^2 = 0,$$

интегрируя по интервалу $(0 < x < p)$, и учитывая краевые условия, получим

$$\frac{d}{c} X_n''^2(p) - \frac{b}{a} X_n''^2(0) + \frac{1}{2} X_n'^2(p) + (\lambda_n - \mu) \int_0^p X_n^2 dx = 0.$$

Так как $ab \leq 0$, $cd \geq 0$, $\lambda_n > 0$, $\mu \leq 0$, то $X_n \equiv 0$. Лемма доказано.

Пусть существует такой номер n^* , что $\Delta(n^*) = 0$, тогда существуют постоянные C_1^*, C_2^*, C_3^* одновременно не все равные нулю, удовлетворяющие системе

$$\begin{cases} C_{1n^*}^* \left(a + bk_{n^*}^2 \right) + C_{2n^*}^* \left(a - \frac{bk_{n^*}^2}{2} \right) - C_{3n^*}^* \frac{\sqrt{3}bk_{n^*}^2}{2} = 0, \\ C_{1n^*}^* e^{k_{n^*} p} \left(c + dk_{n^*}^2 \right) + C_{2n^*}^* e^{-\frac{1}{2}k_{n^*} p} \left(c\alpha_{n^*} + dk_{n^*}^2 \gamma_{n^*} \right) + C_{3n^*}^* e^{-\frac{1}{2}k_{n^*} p} \left(c\beta_{n^*} + dk_{n^*}^2 \delta_{n^*} \right) = 0, \\ k_{n^*} C_{1n^*}^* - \frac{1}{2} k_{n^*} C_{2n^*}^* + \frac{\sqrt{3}}{2} k_{n^*} C_{3n^*}^* = 0. \end{cases}$$

Отсюда имеем, что функция

$$X_{n^*}(x) = C_{1n^*}^* e^{k_{n^*} x} + e^{-\frac{1}{2}k_{n^*} x} \left(C_{2n^*}^* \cos \frac{\sqrt{3}}{2} k_{n^*} x + C_{3n^*}^* \sin \frac{\sqrt{3}}{2} k_{n^*} x \right),$$

является решением краевой задачи (13), но по доказанной лемме должно быть

$$C_{1n^*}^* e^{k_{n^*} x} + e^{-\frac{1}{2}k_{n^*} x} \left(C_{2n^*}^* \cos \frac{\sqrt{3}}{2} k_{n^*} x + C_{3n^*}^* \sin \frac{\sqrt{3}}{2} k_{n^*} x \right) \equiv 0,$$

но это невозможно в силу линейно-независимости функций

$$e^{k_{n^*} x}, \quad e^{-\frac{1}{2}k_{n^*} x} \cos \frac{\sqrt{3}}{2} k_{n^*} x, \quad e^{-\frac{1}{2}k_{n^*} x} \sin \frac{\sqrt{3}}{2} k_{n^*} x.$$

Отсюда функция:

$$\begin{aligned} u^*(x, y) = & \sqrt{\frac{2}{q}} \sum_{n^*=1}^{\infty} \left[C_{1n^*}^* e^{k_{n^*} x} + e^{-\frac{1}{2}k_{n^*} x} \left(C_{2n^*}^* \cos \frac{\sqrt{3}}{2} k_{n^*} x + C_{3n^*}^* \sin \frac{\sqrt{3}}{2} k_{n^*} x \right) \right] \times \\ & \times \cos \frac{(2n^*-1)\pi}{2q} y, \end{aligned}$$

является нетривиальным решением задачи A_1 , а это противоречит теореме единственности.

Итак $\Delta(n) \neq 0$, $n \in N$. Тогда система уравнений (12) имеет решение в виде

$$\begin{aligned}
 C_{1n} &= \frac{2e^{-k_n p}}{\sqrt{3}\Delta} \left[\frac{\psi_{1n}}{k_n^2} e^{-\frac{k_n}{2}p} \left(\frac{c}{k_n^2} \cos \left(\frac{\sqrt{3}}{2} k_n p - \frac{\pi}{6} \right) + d \cos \left(\frac{\sqrt{3}}{2} k_n p + \frac{\pi}{6} \right) \right) - \right. \\
 &\quad - \frac{\sqrt{3}}{2} \frac{\psi_{2n}}{k_n^2} \left(\frac{a}{k_n^2} - b \right) + \frac{\psi_{3n} e^{-\frac{k_n}{2}p}}{k_n} \left\{ \left(\frac{a}{k_n^2} - \frac{1}{2} b \right) \left(\frac{c}{k_n^2} - \frac{1}{2} d \right) \sin \frac{\sqrt{3}}{2} k_n p + \right. \\
 &\quad \left. \left. + \frac{\sqrt{3}}{2} \left(\frac{ad}{k_n^2} - bd + \frac{bc}{k_n^2} \right) \cos \frac{\sqrt{3}}{2} k_n p \right\} \right], \\
 C_{2n} &= \frac{2}{\sqrt{3}\Delta} \left[\frac{\psi_{1n}}{k_n^2} \left\{ e^{-\frac{3k_n}{2}p} \left(\frac{c}{k_n^2} \sin \frac{\sqrt{3}}{2} k_n p + d \sin \left(\frac{\sqrt{3}}{2} k_n p - \frac{2\pi}{3} \right) \right) - \frac{\sqrt{3}}{2} \left(\frac{c}{k_n^2} + d \right) \right\} + \right. \\
 &\quad + \frac{\sqrt{3}}{2} \frac{\psi_{2n}}{k_n^2} e^{-k_n p} \left(\frac{a}{k_n^2} + 2b \right) - \frac{\psi_{3n}}{k_n} \left\{ e^{-\frac{3k_n}{2}p} \left(\frac{a}{k_n^2} + b \right) \left(\frac{c}{k_n^2} \sin \frac{\sqrt{3}}{2} k_n p + \right. \right. \\
 &\quad \left. \left. + d \sin \left(\frac{\sqrt{3}}{2} k_n p - \frac{2\pi}{3} \right) \right) + \frac{\sqrt{3}}{2} b \left(\frac{c}{k_n^2} + d \right) \right\} \right], \\
 C_{3n} &= \frac{2}{\sqrt{3}\Delta} \left[-\frac{\psi_{1n}}{k_n^2} \left\{ \frac{c}{2k_n^2} + \frac{d}{2} - e^{-\frac{3k_n}{2}p} \left(\frac{c}{k_n^2} \cos \frac{\sqrt{3}}{2} k_n p + d \cos \left(\frac{\sqrt{3}}{2} k_n p - \frac{2\pi}{3} \right) \right) \right\} + \right. \\
 &\quad + \frac{3a}{2} \frac{\psi_{2n}}{k_n^4} e^{-k_n p} + \frac{\psi_{3n}}{k_n} \left\{ e^{-\frac{3k_n}{2}p} \left(\frac{a}{k_n^2} + \frac{b}{2} \right) \left(\frac{c}{k_n^2} \cos \frac{\sqrt{3}}{2} k_n p + \right. \right. \\
 &\quad \left. \left. + d \cos \left(\frac{\sqrt{3}}{2} k_n p - \frac{2\pi}{3} \right) \right) - \left(\frac{a}{k_n^2} - \frac{b}{2} \right) \left(\frac{c}{k_n^2} + d \right) \right\} \right],
 \end{aligned}$$

Тогда решение задачи A_1 записывается в следующем виде:

$$u(x, y) = \sqrt{\frac{2}{q}} \sum_{n=1}^{+\infty} \left[C_{1n} e^{k_n x} + e^{-\frac{1}{2} k_n x} \left(C_{2n} \cos \frac{\sqrt{3}}{2} k_n x + C_{3n} \sin \frac{\sqrt{3}}{2} k_n x \right) \right] \cos \frac{(2n-1)\pi}{2q} y. \quad (14)$$

Теперь докажем абсолютную и равномерную сходимость ряда (14) в области $\overline{D}$. Из (14) имеем

$$\begin{aligned}
 |u(x, y)| &= \sqrt{\frac{2}{q}} \sum_{n=1}^{+\infty} \left| \left[C_{1n} e^{k_n x} + e^{-\frac{1}{2} k_n x} \left(C_{2n} \cos \frac{\sqrt{3}}{2} k_n x + C_{3n} \sin \frac{\sqrt{3}}{2} k_n x \right) \right] \right| \times \\
 &\quad \times \left| \cos \frac{(2n-1)\pi}{2q} y \right| \leq M \sum_{n=1}^{+\infty} [|C_{1n}| e^{k_n x} + |C_{2n}| + |C_{3n}|].
 \end{aligned} \quad (15)$$

Найдем оценки C_{in} , $i = \overline{1, 3}$, получим

$$\begin{aligned} |C_{1n}| &\leq M e^{-k_n p} \left[\frac{|\psi_{1n}|}{k_n^2} + \frac{|\psi_{2n}|}{k_n^2} + \frac{|\psi_{3n}|}{k_n} \right], \\ |C_{2n}| &\leq M \left[\frac{|\psi_{1n}|}{k_n^2} + \frac{|\psi_{2n}|}{k_n^2} + \frac{|\psi_{3n}|}{k_n} \right], \\ |C_{3n}| &\leq M \left[\frac{|\psi_{1n}|}{k_n^2} + \frac{|\psi_{2n}|}{k_n^2} + \frac{|\psi_{3n}|}{k_n} \right], \end{aligned}$$

Найденные оценки от C_{in} , $i = \overline{1, 3}$, подставляя в (15), имеем

$$|u(x, y)| \leq M \sum_{n=1}^{+\infty} e^{k_n(x-p)} \left[\frac{|\psi_{1n}|}{k_n^2} + \frac{|\psi_{2n}|}{k_n^2} + \frac{|\psi_{3n}|}{k_n} \right] \leq M \sum_{n=1}^{+\infty} \left[\frac{|\psi_{1n}|}{k_n^2} + \frac{|\psi_{2n}|}{k_n^2} + \frac{|\psi_{3n}|}{k_n} \right]. \quad (16)$$

Интегрируя по частям ψ_{in} , $i = \overline{1, 3}$ и принимая во внимание условие (4), получим

$$\psi_{in} = \left(\frac{q}{\pi} \right)^2 \frac{\Psi_{in}}{n^2}, \quad i = \overline{1, 3},$$

где

$$\Psi_{in} = -\sqrt{\frac{2}{q}} \int_0^q \psi_i''(y) \cos \frac{(2n-1)\pi}{2q} y dy.$$

Тогда

$$|\psi_{in}| \leq M \frac{|\Psi_{in}|}{n^2}, \quad i = \overline{1, 3}, \quad M = \text{const} > 0.$$

Учитывая эти оценки, из (16) находим

$$|u(x, y)| \leq M \sum_{n=1}^{+\infty} \left[\frac{|\Psi_{1n}|}{n^{\frac{10}{3}}} + \frac{|\Psi_{2n}|}{n^{\frac{10}{3}}} + \frac{|\Psi_{3n}|}{n^{\frac{8}{3}}} \right] < \infty.$$

Отсюда следует, что ряд (14) сходится абсолютно и равномерно в области $\overline{D}$.

Теперь докажем, что производные ряда (14) входящий в уравнение (1), также сходятся абсолютно и равномерно в области $\overline{D}$. Для этого вычисляем частную производную по y до второго порядка, из (14) получим

$$\begin{aligned} \frac{\partial^2 u}{\partial y^2} = & -\sqrt{\frac{2}{q}} \left(\frac{\pi}{q} \right)^2 \sum_{n=1}^{+\infty} n^2 \left[C_{1n} e^{k_n x} + e^{-\frac{1}{2} k_n x} \left(C_{2n} \cos \frac{\sqrt{3}}{2} k_n x + C_{3n} \sin \frac{\sqrt{3}}{2} k_n x \right) \right] \times \\ & \times \cos \frac{(2n-1)\pi}{2q} y, \end{aligned}$$

Учитывая оценку $u(x, y)$, имеем

$$\left| \frac{\partial^2 u}{\partial y^2} \right| \leq M \sum_{n=1}^{+\infty} \left[\frac{|\Psi_{1n}|}{n^{\frac{4}{3}}} + \frac{|\Psi_{2n}|}{n^{\frac{4}{3}}} + \frac{|\Psi_{3n}|}{n^{\frac{2}{3}}} \right].$$

Используя неравенства Коши - Буняковского и Бесселя, получим

$$\begin{aligned} \left| \frac{\partial^2 u}{\partial y^2} \right| &\leq M \left[\sum_{n=1}^{+\infty} \frac{|\Psi_{1n}|}{n^{\frac{4}{3}}} + \sum_{n=1}^{+\infty} \frac{|\Psi_{2n}|}{n^{\frac{4}{3}}} + \sqrt{\sum_{n=1}^{+\infty} |\Psi_{3n}|^2} \sqrt{\sum_{n=1}^{+\infty} \frac{1}{n^{\frac{4}{3}}}} \right] \leq \\ &\leq M \left[\sum_{n=1}^{+\infty} \frac{|\Psi_{1n}|}{n^{\frac{4}{3}}} + \sum_{n=1}^{+\infty} \frac{|\Psi_{2n}|}{n^{\frac{4}{3}}} + \|\psi_{3n}''(y)\| \sqrt{\sum_{n=1}^{+\infty} \frac{1}{n^{\frac{4}{3}}}} \right] < \infty, \end{aligned}$$

здесь

$$\sum_{n=1}^{+\infty} |\Psi_{3n}|^2 \leq \|\psi_{3n}''(y)\|_{L_2[0,q]}^2.$$

Следовательно, ряд, соответствующий функции $\frac{\partial^2 u}{\partial y^2}$ сходится абсолютно и равномерно в области $\overline{D}$. Абсолютная и равномерная сходимость третьей производной по x ряда (14) следует из $\left| \frac{\partial^3 u}{\partial x^3} \right| = \left| \frac{\partial^2 u}{\partial y^2} \right|$ и доказанного выше.

Теорема 2 доказана.

Задача A_2 . Найти решение уравнения (1) в области D из класса $C_{x,y}^{3,2}(D) \cap C_{x,y}^{2,1}(\overline{D})$, удовлетворяющее следующие краевые условия:

$$\begin{cases} u(x, 0) = u_y(x, q) = 0, & 0 \leq x \leq p, \\ \begin{cases} au(0, y) + bu_{xx}(0, y) = \psi_1(y), \\ cu(p, y) + du_{xx}(p, y) = \psi_2(y), \\ u_x(0, y) = \psi_3(y), \end{cases} & 0 \leq y \leq q, \end{cases}$$

где $\psi_i(y) \in C^2(0, q)$, $i = \overline{1, 3}$ – заданные достаточно гладкие функции, причем

$$\psi_i(0) = \psi_i'(q) = 0, \quad i = \overline{1, 3}.$$

Существование и единственность решения задачи A_2 доказывается так же, как и в задаче A_1 .

V. ВЫВОДЫ

В данной работе в прямоугольной области исследованы краевые задачи для вязко-транзвукового уравнения с несимметричными условиями по времени. Единственность решения поставленных задач доказаны с методом интегралов энергии. Существование решения доказаны методом разделения переменных. Решения поставленных задач найдены в виде бесконечных рядов, доказаны равномерная сходимость и возможность почленного дифференцирования при некоторых условиях на заданные функции. Сходимость производной второго порядка по переменной y доказывается с помощью неравенств Коши - Буняковского и Бесселя. При обосновании равномерной сходимости решения доказывается отсутствие “малого знаменателя”.

Список литературы

1. Юлдашев Т. К. Обратная задача для одного Фредгольма в частных производных третьего интегро-дифференциального уравнения порядка // Вестн. Сам. гос. техн. ун-та. Сер.

Физ.-мат. науки, - Самара, 2014. - № 1(34). - С.56-65.

2. Диесперов В.Н., Ломакин Л.А. Об одной краевой задаче для линеаризованного осесимметрического ВТ уравнения // Журнал вычисл. матем. и матем. физики. - Москва, 1974. - Т. 14. - № 5. - С. 1244-1260.

3. Рыжов О.С., Шефтер Г.М. О влиянии вязкости и теплопроводности на структуру сжимаемых течений // Прикл. Матем. и механ., - Москва, 1964. Т. 28. Вып. 6. - С. 996-1007.

4. Sternberg J. Triple-shock wave intersections // Phys. Fluids, 1956, -Vol. 2, № 2. - pp. 179-206.

5. Sichel M. Leading edge pf a shock-induced boundary layer // Phys. Fluids, 1962, -Vol. 5, - № 10. - pp. 1168-1180.

6. Sichel M. Structure on weak non-Hugoniot shocks // Phys. Fluids, 1963. Vol. 6, № 5. - pp. 653-662.

7. Рыжов О. С. Асимптотическая картина обтекания тел вращения со звуковым потоком вязкого и теплопроводящего газа // Прикл. Матем. и механ., - Москва, 1965. - Т. 29. Вып. 6. - С. 1004-1014.

8. Диесперов В.Н. О функции Грина линеаризованного вязкого трансзвукового уравнения // Журнал вычисл. матем. и матем. физики. - Москва, 1972. - Т. 12. - № 5. - С. 1265-1279.

9. Block H. Sur les equations lineaires aux derives parielles a carateristiques multiples // Ark. Mat. Astron. Fus. Note 1, - 1912, 7(13), - pp. 1-34; Note 2, 1912, ibid. 7(21), - pp. 1-30; Note 3, 1912 - 1913, ibid. 8(23). - pp. 1-51.

10. Del Vecchio E. Sulleequazioni

$$z_{xxx} - z_y + \varphi_1(x, y) = 0,$$

$$z_{xxx} - z_{yy} + \varphi_2(x, y) = 0 // Memorie R. Accad. Sci. Ser.2. - Torino, 1915, 66. - pp. 1-41.

11. Cattabriga L. Potenziali di linea e di dominio per equazioni non paraboliche in due variabilia caratteristiche multiple // Rendiconti del seminario matimatico della univ. di Padava. - 1961, 31. - pp. 1-45.

12. Джураев Т. Д., Апаков Ю.П. Об автомодельном решении одного уравнения$$

третьего порядка с кратными характеристиками // Вестник Сам. гос. техн. ун-та. Сер. Физ.-мат. науки, - Самара, 2007. - № 2(15). - С.18-26.

13. Джураев Т. Д., Апаков Ю.П. К теории уравнения третьего порядка с кратными характеристиками, содержащего вторую производную по времени // Украинский математический журнал. – Киев, 2010, том 62. № 1. - С. 40-51.

14. Абдуназаров С. Краевые задачи для одного неклассического уравнений третьего порядка // Диссертация. Ташкент, 1991. 239 стр.

15. Apakov Y. P., Umarov R. A. Solution of the Boundary Value Problem for a Third Order Equation with Little Terms. Construction of the Green's Function // Lobachevskii Journal of Mathematics. 2022 Vol.43, No 3. pp. 738-748.

16. Apakov Y. P., Umarov R. A. On the third boundary problem for a nonhomogeneous third order equation with multiple characteristics // Uzbek Mathematical Journal. 2023, Volume 67, Issue 1, pp. 4-14.

17. Irgashev B.Y. A Boundary Value Problem with Conjugation Conditions for a Degenerate Equation with the Caputo Fractional Derivative // Russian mathematics. 2022, Volume 66, Issue 4, p. 24-31. <https://doi.org/10.3103/S1066369X2204003X>

18. Апаков Ю.П., Умаров Р.А. О разрешимости одной краевой задачи для уравнения третьего порядка с кратными характеристиками в прямоугольной области // Научный вестник НамГУ. 2020. N 1 С. 6-17.

19. Apakov Y. P., Umarov R. A. Solution of a boundary value problem for a third-order inhomogeneous equation with multiple characteristics with the construction of the Green's function // Bulletin of the Karaganda University. Mathematics Series. 2024. Т. 114. No. 2. pp. 22-39. <https://doi.org/10.31489/2024M2/22-39>.

20. Тихонов А.Н., Самарский А.А. Уравнения математической физики. - М.: «Наука», 1966. - 724 с.

21. Умаров Р.А. Краевые задачи для уравнения третьего порядка с кратными характеристиками с младшими членами // Дисс... док.фил.по физ.-мат. наук. - Фергана: Ферганский государственный университет, 2025. 98 с.

Buziladigan oddiy differensial tenglama uchun nolokal shartli spektral masalalar

Karimov K.T.^{1,2} Toshmatov M.M.³

¹Farg'ona davlat universiteti, Farg'ona, O'zbekiston;

²O'zbekiston Respublikasi Fanlar akademiyasi V.I. Romanovskiy nomidagi

Matematika instituti; karimovk80@mail.ru

³Andijon davlat universiteti, Andijon, O'zbekiston; tmunosibjon@gmail.com

Annotatsiya: Ushbu maqolada buziladigan oddiy differensial tenglama uchun spektral masalalar qo'yilgan va tadqiq qilingan. Qo'yilgan masalalarning sanoqli sondagi xos qiymatlari va ularga mos keluvchi xos funksiyalari topilgan. Barcha masalalarning xos funktsiyalari Gaussning gipergeometrik funksiyasi orqali ifodalangan.

Kalit so'zlar: oddiy differensial tenglama, nolokal shart, xos qiymat, xos funktsiya, spektral masala.

Аннотация: В данной работе сформулирована и исследована спектральные задачи для вырождающегося обыкновенного дифференциального уравнения. Найдено счётное число собственных значений и соответствующие им собственные функции поставленных задач. Система собственных функций всех задач выражается через гипергеометрической функцией Гаусса.

Ключевые слова: обыкновенное дифференциальное уравнение, нелокальное условие, собственное значение, собственная функция, спектральная задача.

Abstract. In this paper, spectral problems for a degenerate ordinary differential equation are formulated and investigated. A countable number of eigenvalues and corresponding eigenfunctions of the problems are found. The system of eigenfunctions of all problems is expressed through the Gauss hypergeometric function.

Key words: ordinary differential equation, nonlocal condition, eigenvalue, eigenfunction, spectral problem.

1. KIRISH

λ parametrning shunday qiymatlari topilsinki, bu qiymatlarda

$$L[y] = -\lambda \rho(x)y(x), \quad x \in (a, b) \quad (A)$$

differensial tenglamaning $[a, b]$ segmentda aniqlangan, uzliksiz va chegaraviy

$$E_m[y] = 0, \quad m = \overline{1, n} \quad (B)$$

chegaraviy shartlarni qanoatlantiruvchi trivial bo'lmagan yechimi mavjud bo'lsin, bu yerda λ -sonli parametr, $\rho(x)$ -berilgan funktsiya, $y(x)$ -noma'lum funktsiya.

Odatda bunday tarzda qo'yilgan masalalar spektral masalalar deb ataladi. $\{(A), (B)\}$ masalaning trivial bo'lmagan yechimlari mavjud bo'lgan λ ning qiymatlari, bu masalaning xos qiymatlari(sonlari) deb, bu qiymatlarga mos trivialmas yechimlar esa xos funktsiyalari deb ataladi.

Ma'lumki (A) ko'rinishidagi differensial tenglamalar uchun chegaraviy masalalar o'rganishda chegaraviy shartlar nafaqat (B) ko'rinishida, balki boshqa ko'rinishlarda ham berilishi mumkin. Shunga mos ravishda spektral masalalarda ham chegaraviy shartlar (B) dan boshqacharoq berilishi mumkin.

2. MASALANING QO'YILISHI VA ASOSIY NATIJA

Biz bu ishda buziladigan oddiy differensial tenglama uchun nolokal shartli spektral masalalarni tadqiq qilamiz.

A₁ masala. Quyidagi

$$S''(\varphi) + 4\beta \operatorname{ctg} 2\varphi S'(\varphi) + \mu S(\varphi) = 0, \quad (1)$$

tenglamaning

$$qS(0) = S\left(\frac{\pi}{2}\right),$$

$$q \lim_{\varphi \rightarrow 0} (\sin^{2\beta} \varphi) S'(\varphi) \lim_{\varphi \rightarrow \frac{\pi}{2}} (\cos^{2\beta} \varphi) S'(\varphi) \quad (2)$$

nolokal chegaraviy shartlarni qanoatlantiruvchi trivial bo'lmagan yechimi topilsin, bu yerda $\beta, q \in \mathbb{R}$, jumladan $\beta \in (0, 1/2)$.

A₂ masala. (1) tenglamaning

$$qS(0) = S\left(\frac{\pi}{2}\right),$$

$$\lim_{\varphi \rightarrow \frac{\pi}{2}} (\cos^{2\beta} \varphi) S'(\varphi) = 0 \quad (3)$$

nolokal chegaraviy shartlarni qanoatlantiruvchi trivial bo'lmagan yechimi topilsin.

A₃ masala. (1) tenglamaning

$$qS(0) = S\left(\frac{\pi}{2}\right),$$

$$\lim_{\varphi \rightarrow 0} (\sin^{2\beta} \varphi) S'(\varphi) = 0 \quad (4)$$

nolokal chegaraviy shartlarni qanoatlantiruvchi trivial bo'lmagan yechimi topilsin.

A4 masala. (1) tenglamaning

$$S(0) = 0, \quad q \lim_{\varphi \rightarrow 0} (\sin^{2\beta} \varphi) S'(\varphi) = \lim_{\varphi \rightarrow \frac{\pi}{2}} (\cos^{2\beta} \varphi) S'(\varphi) \quad (5)$$

nolakal chegaraviy shartlarni qanoatlantiruvchi trivial bo'lmagan yechimi topilsin.

A5 masala. (1) tenglamaning

$$S\left(\frac{\pi}{2}\right) = 0, \quad q \lim_{\varphi \rightarrow 0} (\sin^{2\beta} \varphi) S'(\varphi) = \lim_{\varphi \rightarrow \frac{\pi}{2}} (\cos^{2\beta} \varphi) S'(\varphi) \quad (6)$$

nolakal chegaraviy shartlarni qanoatlantiruvchi trivial bo'lmagan yechimi topilsin.

Birinchi bo'lib, A₁ masalani tadqiq qilamiz.

(1) tenglamaning umumiy yechimi [1] ishda topilgan bo'lib u quyidagi ko'rinishga ega:

$$S(\varphi) = c_1 F\left(\beta - \frac{\omega}{2}, \beta + \frac{\omega}{2}; \beta + \frac{1}{2}; \sin^2 \varphi\right) + c_2 (\sin^2 \varphi)^{\frac{1}{2}-\beta} F\left(\frac{1-\omega}{2}, \frac{1+\omega}{2}; \frac{3}{2} - \beta; \sin^2 \varphi\right), \quad (7)$$

bu yerda c_1 va c_2 ixtiyoriy o'zgarmas sonlar, $F(\dots)$ esa Gaussning gipergeometrik funksiyasi [2], $\omega = \sqrt{\mu + 4\beta^2}$.

(7) funksiyani (2) shartning birinchisiga bo'ysundiramiz va ma'lum bo'lgan [2]

$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad c-a-b > 0$$

formulani e'tiborga olib quyidagiga ega bo'lamiz:

$$c_1 \left(\frac{\Gamma(\beta + \frac{1}{2})\Gamma(\frac{1-\omega}{2})}{\Gamma(\frac{1+\omega}{2})\Gamma(\frac{1-\omega}{2})} - q \right) + c_2 \frac{\Gamma(\frac{3}{2}-\beta)\Gamma(\frac{1}{2}-\beta)}{\Gamma(1-\beta+\frac{\omega}{2})\Gamma(1-\beta-\frac{\omega}{2})} = 0. \quad (8)$$

Endi (7) funksiyaning hosilasini hisoblaymiz:

$$S'(\varphi) = c_1 \frac{\beta^2 - \frac{\omega^2}{4}}{\frac{1}{2} + \beta} \sin 2\varphi F\left(1 + \beta - \frac{\omega}{2}, 1 + \beta + \frac{\omega}{2}; \frac{3}{2} + \beta; \sin^2 \varphi\right) + c_2 \left(\frac{1}{2} - \beta\right) \sin 2\varphi (\sin^2 \varphi)^{-\frac{1}{2}-\beta} F\left(\frac{1-\omega}{2}, \frac{1+\omega}{2}; \frac{1}{2} - \beta; \sin^2 \varphi\right).$$

Bu ifodani Gauss gipergeometrik funksiyasi uchun o'rinli bo'lgan

$$F(a, b; c; x) = (1-x)^{c-a-b} F(c-a, c-b; c; x)$$

avtotransformatsiya formulasini [2] qo'llab, quyidagicha yozishimiz mumkin:

$$S'(\varphi) = c_1 \frac{4\beta^2 - \omega^2}{2\beta + 1} \sin \varphi \cos^{-2\beta} \varphi F\left(\frac{1+\omega}{2}, \frac{1-\omega}{2}; \frac{3}{2} + \beta; \sin^2 \varphi\right) + c_2 (1-2\beta) \sin^{-2\beta} \varphi \cos^{-2\beta} \varphi F\left(\frac{\omega}{2} - \beta, -\frac{\omega}{2} - \beta; \frac{1}{2} - \beta; \sin^2 \varphi\right). \quad (9)$$

(9) ni (2) shartning ikkinchisiga bo'ysundirib, quyidagiga ega bo'lamiz:

$$\lim_{\varphi \rightarrow 0} q (\sin^{2\beta} \varphi) S'(\varphi) = c_2 q (1-2\beta),$$

$$\lim_{\varphi \rightarrow \frac{\pi}{2}} (\cos^{2\beta} \varphi) S'(\varphi) = c_1 \frac{4\beta^2 - \omega^2}{2\beta + 1} F\left(1 + \frac{\omega}{2}, 1 - \frac{\omega}{2}; \beta + \frac{3}{2}; 1\right) + c_2 (1-2\beta) F\left(\frac{\omega}{2} - \beta, -\frac{\omega}{2} - \beta; \frac{1}{2} - \beta; 1\right) =$$

$$+ c_1 \frac{4\beta^2 - \omega^2}{2\beta + 1} \frac{\Gamma(\beta + \frac{3}{2})\Gamma(\beta + \frac{1}{2})}{\Gamma(1 + \beta - \frac{\omega}{2})\Gamma(1 + \beta + \frac{\omega}{2})} + c_2 (1-2\beta) \frac{\Gamma(\frac{1}{2} - \beta)\Gamma(\frac{1}{2} + \beta)}{\Gamma(\frac{1-\omega}{2})\Gamma(\frac{1+\omega}{2})},$$

$$2c_1 \frac{\Gamma(\beta + \frac{1}{2})\Gamma(\beta + \frac{1}{2})}{\Gamma(\beta - \frac{\omega}{2})\Gamma(\beta + \frac{\omega}{2})} + c_2 (1-2\beta) \left(\frac{\Gamma(\frac{1}{2}-\beta)\Gamma(\frac{1}{2}+\beta)}{\Gamma(\frac{1-\omega}{2})\Gamma(\frac{1+\omega}{2})} - q \right) = 0.$$

(10)

Natijada bizda c_1 va c_2 ga nisbatan {(8),(10)} tenglamalar sistemasi hosil bo'ldi. Bu sistemani yechishda quyidagi

$$\Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin a\pi}$$

ma'lum formuladan [2] foydalanganimiz va $q=1$ deb olinganda

$$\sin \frac{(\frac{\omega}{2} + \beta)\pi}{2} \sin \frac{(\frac{\omega}{2} - \beta)\pi}{2} = 0$$

trigonometrik tenglama paydo bo'ladi.

Bu trigonometrik tenglamani yechib

$$\omega_n^{(1)} = (4n - 2\beta),$$

$\omega_n^{(2)} = (4n + 2\beta)$, $n \in N$ ni hosil qilamiz so'ngra

$\omega = \sqrt{\mu + 4\beta^2}$ formuladan foydalanib, qo'yilgan nolokal masalaning xos qiymatlarini $\mu_n^{(1)} = 16n^2 - 16\beta n$,

$$\mu_n^{(2)} = 16n^2 + 16\beta n, n \in N$$

ko'rinishida yozib olamiz. ω_n ning ko'rinishini inobatga olib, μ_n xos qiymatlarga mos xos funksiyalarni quyidagi ko'rinishda hosil qilamiz:

$$\begin{aligned} S_n^j(\varphi) &= c_1 F\left(\beta - \frac{\omega_n}{2}, \beta + \frac{\omega_n}{2}; \beta + \frac{1}{2}; \sin^2 \varphi\right) \\ &+ c_2 (\sin^2 \varphi)^{\frac{1}{2}-\beta} F\left(\frac{1-\omega_n}{2}, \frac{1+\omega_n}{2}; \frac{3}{2} - \beta; \sin^2 \varphi\right), \quad j = \overline{1,2} \end{aligned}$$

Xuddi shunday usul yordamida qolgan masalalarning xos qiymatlari va xos funksiyalari quydagicha topilgan.

A2 masala uchun:

$$\begin{aligned} \mu_n^{(1)} &= 16n^2 - 16\beta n, \\ \mu_n^{(2)} &= 16n^2 + 16\beta n, \quad n \in N, \\ \omega_n^{(j)} &= \sqrt{\mu_n^{(j)} + 4\beta^2}, \\ S_n^{(j)}(\varphi) &= \end{aligned}$$

$$\begin{aligned} &c_1 F\left(\beta - \frac{\omega_n^{(j)}}{2}, \beta + \frac{\omega_n^{(j)}}{2}; \beta + \frac{1}{2}; \sin^2 \varphi\right) + \\ &c_2 (\sin^2 \varphi)^{\frac{1}{2}-\beta} F\left(\frac{1-\omega_n^{(j)}}{2}, \frac{1+\omega_n^{(j)}}{2}; \frac{3}{2} - \beta; \sin^2 \varphi\right), \quad j = \overline{1,2}. \end{aligned}$$

A3 masala uchun:

$$\begin{aligned} \mu_n^{(1)} &= 16n^2 - 16\beta n, \\ \mu_n^{(2)} &= 16n^2 + 16\beta n, \quad n \in N, \\ S_n^{(j)}(\varphi) &= c_1 F\left(\beta - \frac{\omega_n^{(j)}}{2}, \beta + \frac{\omega_n^{(j)}}{2}; \beta + \frac{1}{2}; \sin^2 \varphi\right), \quad j = \overline{1,2}. \end{aligned}$$

A4 masala uchun:

$$\begin{aligned} \mu_n^{(1)} &= 16n^2 - 16\beta n, \\ \mu_n^{(2)} &= 16n^2 + 16\beta n, \quad n \in N \\ S_n^{(j)}(\varphi) &= c_2 (\sin^2 \varphi)^{\frac{1}{2}-\beta} F\left(\frac{1-\omega_n^{(j)}}{2}, \frac{1+\omega_n^{(j)}}{2}; \frac{3}{2} - \beta; \sin^2 \varphi\right), \quad j = \overline{1,2}. \end{aligned}$$

A5 masala uchun:

$$\begin{aligned} \mu_n^{(1)} &= 16n^2 - 16\beta n, \\ \mu_n^{(2)} &= 16n^2 + 16\beta n, \quad n \in N \\ S_n^{(j)}(\varphi) &= c_1 F\left(\beta - \frac{\omega_n^{(j)}}{2}, \beta + \frac{\omega_n^{(j)}}{2}; \beta + \frac{1}{2}; \sin^2 \varphi\right) + \\ &c_2 (\sin^2 \varphi)^{\frac{1}{2}-\beta} F\left(\frac{1-\omega_n^{(j)}}{2}, \frac{1+\omega_n^{(j)}}{2}; \frac{3}{2} - \beta; \sin^2 \varphi\right), \quad j = \end{aligned}$$

ADABIYOTLAR

1. Салахитдинов М.С., Уринов А.К. К спектральной теории уравнений смешанного типа. –Ташкент: Mumtoz So'z. –2010. -355 с.

2. Бейтмен Г., Эрдейи А. Высшие трансцендентные функции. Гипергеометрические функции. Лежандра.-М.:Наука.-1973.-296 с.

Решение краевой задачи для уравнения третьего порядка с переменными коэффициентами

Апаков Юсупжон Пулатович^{1,2,3}

Умаров Рахматилла Акрамович^{4,5}

¹Институт математики им. В.И. Романовского АН РУз,

²Наманганский государственный технический университет

E-mail: ³yusupjonapakov@gmail.com

⁴Андижанский институт сельского хозяйства и агротехнологии

E-mail: ⁵r.umarov1975@mail.ru

Аннотация. В работе рассматривается несимметричная краевая задача по времени в прямоугольной области для неоднородного уравнения третьего порядка с младшими членами. Единственность решения поставленной задачи доказана методом интегральных тождеств. Решение задачи построено в виде суммы ряда по системе собственных функции. Приведено обоснование сходимости построенного в классе регулярных решений.

Ключевые слова: Дифференциальное уравнение, третий порядок, кратные характеристики, несимметричное условие, регулярное решение, единственность, существование, функция Грина.

Annotatsiya. Maqolada uchinchi tartibli kichik tartibli hosilalarga ega bo'lgan bir jinsli bo'lmagan tenglama uchun to'rtburchaklar sohada vaqt bo'yicha simetrik bo'lmagan chegaraviy masala ko'rib chiqiladi. Masala yechimini yagonaligi inergiya integrallari usulida isbotlangan. Masala yechimi xos funksiyalar sistemasi yig'indisi ko'rinishida topilgan. Topilgan yechimni yaqinlashuvchiligi berilgan sinfda isbotlangan.

Kalit so'zlar: Differensial tenglama, uchinchi tartib, karrali xarakteristikali tenglama, simetrik bo'lmagan shart, regular yechim, yagonalik, mavjudlik, Grin funktsiyasi.

Abstract. The paper considers an asymmetric boundary value problem in time in a rectangular domain for a third-order inhomogeneous equation with lower terms. The uniqueness of the solution to the problem is proved by the method of integral identities. The solution to the problem is constructed as a sum of a series in a system of eigenfunctions. The convergence of the constructed solution in the class of regular solutions is substantiated.

Keywords: Differential equation, third order, multiple characteristics, asymmetric condition, regular solution, uniqueness, existence, Green's function.

1. Введение

Дифференциальные уравнения в частных производных третьего порядка рассматривается при решении задач теории нелинейной акустики и в гидродинамической теории космической плазмы, фильтрации жидкости в пористых средах [1].

В работе [2], учитывая свойства вязкости и теплопроводности газа, из системы Навье-Стокса было получено уравнение третьего порядка с кратными характеристиками, содержащее вторую производную по времени, $t = y$

$$u_{xxx} + u_{yy} - \frac{\nu}{y} u_y = u_x u_{xx}, \quad \nu = const.$$

Это уравнение при $\nu = 1$ описывает осесимметричный поток, а при $\nu = 0$ описывает плоско - параллельный поток [3].

Первые результаты по уравнению третьего порядка с кратными характеристиками были получены в работах Н. Block [4], E. DelVecchio[5].

L. Catabriga в работе [6] для уравнения $D_x^{2n+1}u - D_y^2u = 0$ построил фундаментальное решение в виде двойного несобственного интеграла и изучил свойства потенциала, решил краевые задачи.

В работах [7-8] построены фундаментальные решения уравнения третьего порядка с кратными характеристиками, содержащие вторые производные по времени, выраженные через вырожденные гипергеометрические функции, изучены их свойства, найдена оценки при больших значениях времени.

В работах [9-14], рассмотрены краевые задачи для уравнений третьего

порядка с кратными характеристиками, при помощи построения функции Грина.

В работах [15-18] исследуются краевые задачи для уравнений смешанного типа третьего порядка, при этом разработанная методика была существенно использована в этих исследованиях. Также,

отметим работы [19-22], в которых, рассмотрены краевые задачи для уравнений третьего порядка. В работах [23-25] найдены решения краевых задачи с симметричными условиями по времени для уравнений третьего порядка с постоянными коэффициентами.

2. Постановка задачи и единственность решения

В области $D = \{(x, y) : 0 < x < p, 0 < y < q\}$ рассмотрим следующее уравнение третьего порядка в виде

$$L(u) = U_{xxx} - U_{yy} + A_1(x)U_{xx} + A_2(x)U_x + A_3(x)U + A_4U_y = g_1(x, y), \quad (1)$$

где $p, q, A_i \in R, A_i(x), i = \overline{1, 3}, g_1(x, y)$ заданные достаточно гладкие функции в $\bar{D}$.

Заменой

$$U(x, y) = \exp\left(-\frac{1}{3} \int_0^x A_1(\xi) d\xi + \frac{A_4}{2} y\right) u(x, y),$$

уравнение (1) можно привести к виду

$$u_{xxx} - u_{yy} + a_1(x)u_x + a_2(x)u = g(x, y), \quad (2)$$

где

$$a_1(x) = -A_1'(x) - \frac{1}{3}A_1^2(x) + A_2(x),$$

$$a_2(x) = -\frac{1}{3}A_1''(x) + \frac{2}{27}A_1^3(x) - \frac{1}{3}A_1(x)A_2(x) + A_3(x) + \frac{A_4^2}{4},$$

$$g(x, y) = \exp\left(\frac{1}{3} \int_0^x A_1(\xi) d\xi - \frac{A_4}{2} y\right) g_1(x, y).$$

Задача А. Найти функцию $u(x, y)$ из класса $C_{x,y}^{3,2}(D) \cap C_{x,y}^{2,1}(\bar{D})$, удовлетворяющую уравнению (2) и следующим краевым условиям:

$$u(x, 0) = 0, \quad u_y(x, q) = 0, \quad 0 \leq x \leq p, \quad (3)$$

$$u_{xx}(0, y) = \psi_1(y), \quad u(p, y) = \psi_2(y), \quad u_x(p, y) = \psi_3(y), \quad 0 \leq y \leq q, \quad (4)$$

где $\psi_i(y), x \in [0, p], i = \overline{1, 3}, g(x, y)$ заданные достаточно гладкие функции.

Теорема 1. Если задача А имеет решение, то при выполнении условий $a_1(0) \leq 0, a_2(x) - \frac{1}{2}a_1'(x) \geq 0$, оно единственно.

Доказательство. Предположим, обратное. Пусть задача А имеет два решения $u_1(x, y)$ и $u_2(x, y)$. Тогда функция $u(x, y) = u_1(x, y) - u_2(x, y)$ удовлетворяет однородному уравнению (2) с однородными краевыми условиями. Докажем, что $u(x, y) \equiv 0$ в $\bar{D}$.

В области D справедливо тождество

$$uL[u] = uu_{xxx} - uu_{yy} + a_1(x)uu_x + a_2(x)u^2 = 0,$$

или

$$\frac{\partial}{\partial x} \left(uu_{xx} - \frac{1}{2}u_x^2 + \frac{1}{2}a_1(x)u^2 \right) - \frac{\partial}{\partial y} (uu_y) + u_y^2 + \left(a_2(x) - \frac{1}{2}a_1'(x) \right) u^2 = 0.$$

Интегрируем это тождество в области D учитывая однородные краевые условия, получим

$$-\frac{1}{2}a_1(0)\int_0^q u^2(0,y)dy + \frac{1}{2}\int_0^q u_x^2(0,y)dy + \iint_D u_y^2 dx dy + \iint_D \left(a_2(x) - \frac{1}{2}a_1'(x)\right) u^2(x,y) dx dy = 0,$$

Если $a_2(x) - \frac{1}{2}a_1'(x) > 0$, из четвертого слагаемого, получим $u(x,y) \equiv 0$, $(x,y) \in \bar{D}$. Если,

$a_2(x) - \frac{1}{2}a_1'(x) = 0$ тогда из третьего слагаемого $u_y(x,y) = 0$. Отсюда $u(x,y) = f(x)$.

Учитывая краевые условие (3), получим $u(x,0) = f(x) = 0$. Тогда имеем $u(x,y) \equiv 0$.

Теорема доказана.

Замечание. Отметим, что при нарушении условий теоремы 1, т.е. $a_1(0) > 0$, $a_2(x) < 0$ однородная задача А для однородного уравнения (2) может иметь нетривиальное решение. Например, уравнение

$$u_{xxx}(x,y) + \left(\frac{(2k+1)\pi}{2p}\right)^2 u_x(x,y) - \left(\frac{\pi(2n-1)}{2q}\right)^2 u(x,y) - u_{yy}(x,y) = 0,$$

с однородными условиями (3)-(4) имеет нетривиальное решение

$$u(x,y) = \left(1 + (-1)^{k+1} \sin\left(\frac{(2k+1)\pi}{2p}x\right)\right) \sin\left(\frac{\pi(2n-1)}{2q}y\right), \quad n, k \in N.$$

3. Существование решения

Теорема 2. Если выполняются следующие условия:

$$1) g(x,0) = g_\eta(x,q) = 0, \quad \frac{\partial^2 g(x,y)}{\partial x \partial y} \in C[0,q], \quad 0 \leq x \leq p;$$

$$2) \psi_i(0) = \psi_i'(q) = \psi_i''(0) = 0, \quad \psi_i(0) = \psi_i'(q) = \psi_i''(0) = 0;$$

$$3) 0 \leq Q < \frac{\lambda_1^2}{Kp(1+\lambda_1)};$$

$$4) a_1(0) = 0,$$

то решение задачи А существует. Здесь $C = \max\{|a_1|, |a_2|\}$, $\lambda_1 = \sqrt[3]{\left(\frac{\pi}{q}\right)^2}$,

$$K = \frac{16}{3} \left(1 - \exp\left(-\frac{2\sqrt{3}\pi}{3}\right)\right)^{-1}.$$

Доказательство. Решение задачи А ищем в виде

$$u(x,y) = \sum_{n=1}^{\infty} X_n(x) Y_n(y). \quad (6)$$

Поставляя (6) в уравнение (2) по переменной $Y_n(y)$, получим следующую задачу Штурма-Лиувилля:

$$\begin{cases} Y_n'' + \lambda_n^3 Y_n = 0, \\ Y_n(0) = Y_n'(q) = 0. \end{cases} \quad (7)$$

Собственные значения задачи (7) имеет вид

$$\lambda_n^3 = \left(\frac{\pi(2n-1)}{2q}\right)^2, \quad n = 1, 2, \dots \quad (8)$$

Соответствующая собственная функция задачи (7) имеет вид:

$$Y_n(y) = \sqrt{\frac{2}{q}} \sin\left(\frac{\pi(2n-1)}{2q} y\right) \quad (9)$$

Известно [26], что ортонормированная система функций (9) полна в $L_2(0, q)$.

Разложим $g(x, y)$ в ряд Фурье по $\{Y_n(y)\}$:

$$g(x, y) = \sum_{n=1}^{\infty} g_n(x) Y_n(y),$$

где $g_n(x) = \int_0^q g(x, \eta) Y_n(\eta) d\eta$. Интегрируем функцию $g_n(x)$ по частям два раза и учитывая условия 1 теоремы 2 имеем оценку

$$|g_n(x)| \leq M \frac{|\bar{g}_n(x)|}{n^2},$$

здесь $\bar{g}_n(x) = \int_0^q g_\eta(x, \eta) Y_n(\eta) d\eta$, $M - \text{const}$.

В дальнейшем для всех найденных известных положительных постоянных введем обозначение одной буквой M .

По переменной $X_n(x)$, получим следующую задачу:

$$\begin{cases} X_n''' + a_1(x) X_n' + a_2(x) X_n + \lambda_n^3 X_n = g_n(x), \\ X_n''(0) = \psi_{1n}, X_n(p) = \psi_{2n}, X_n'(p) = \psi_{3n}, \end{cases} \quad (10)$$

где $\psi_{in} = \int_0^q \psi_i(\eta) Y_n(\eta) d\eta$, $i = \overline{1, 3}$. Учитывая условие 2 теоремы 2 и интегрируя по частям ψ_{in} три раза, находим оценку:

$$|\psi_{in}| \leq M \frac{|\Psi_{in}|}{n^3}, i = \overline{1, 3}, \quad (11)$$

где $\Psi_{in} = \int_0^q \psi_i'''(\eta) \cos \frac{\pi n}{q} \eta d\eta$.

С помощью функции

$$V_n(x) = X_n(x) - \rho_n(x) \quad (12)$$

изменим граничные условия на однородные.

Функция $\rho(x)$ имеет вид:

$$\rho_n(x) = \psi_{2n} - \psi_{3n} p + \frac{\psi_{1n}}{2} p^2 + (\psi_{3n} - \psi_{1n} p) x + \frac{\psi_{1n}}{2} x^2. \quad (13)$$

Подставляя (12) в (10) получим задачу

$$\begin{cases} V''' + \lambda_n^3 V = \lambda_n^3 f_n(x) - a_1(x) V' - a_2(x) V, \\ V''(0) = V(p) = V'(p) = 0, \end{cases} \quad (14)$$

здесь

$$\begin{aligned} f_n(x) = & \left(\frac{a_1(x)p - a_1(x)x + a_2(x)px}{\lambda_n^3} - \frac{a_2(x)p^2 + a_2(x)x^2}{2\lambda_n^3} - \frac{p^2 + x^2}{2} + px \right) \psi_{1n} - \\ & - \left(\frac{a_2(x)}{\lambda_n^3} + 1 \right) \psi_{2n} + \left(\frac{a_2(x)p - a_1(x) - a_2(x)x}{\lambda_n^3} + p - x \right) \psi_{3n} + \frac{g_n(x)}{\lambda_n^3}. \end{aligned}$$

Из $f_n(x)$, учитывая $\lambda_n^3 = \left(\frac{\pi n}{q}\right)^2$, имеем оценку:

$$\begin{aligned} |f_n(x)| &\leq M|\psi_{1n}| + M|\psi_{2n}| + M|\psi_{3n}| + M\frac{\bar{g}_n(x)}{n^2 \cdot \lambda_n^3} \leq \\ &\leq M\left(\frac{|\Psi_{1n}|}{n^3} + \frac{|\Psi_{2n}|}{n^3} + \frac{|\Psi_{3n}|}{n^3} + \frac{\bar{g}_n(x)}{n^4}\right) \leq \frac{M}{n^3}\left(|\Psi_{1n}| + |\Psi_{2n}| + |\Psi_{3n}| + \frac{1}{n}|\bar{g}_n(x)|\right). \end{aligned}$$

Остальные оценки получим аналогично, тогда имеем

$$|f'_n(x)| \leq \frac{M}{n^3}\left(|\Psi_{1n}| + |\Psi_{2n}| + |\Psi_{3n}| + \frac{1}{n}|\bar{g}'_n(x)|\right). \quad (15)$$

Задача (14) эквивалентна интегро-дифференциальному уравнению

$$\begin{aligned} V_n(x) &= \lambda_n^3 \int_0^p G_n(x, \xi) f_n(\xi) d\xi - \int_0^p G_n(x, \xi) a_1(\xi) V'_n(\xi) d\xi - \\ &\quad - \int_0^p G_n(x, \xi) a_2(\xi) V_n(\xi) d\xi, \end{aligned} \quad (16)$$

где $G_n(x, \xi)$ функция Грина задачи (14), которая обладает следующими свойствами:

$$\begin{aligned} \frac{\partial^3 G_n(x, \xi)}{\partial x^3} + \lambda_n^3 G_n(x, \xi) &= 0 \\ G_{1nxx}(0, \xi) = G_{2n}(p, \xi) = G_{2nxx}(p, \xi) &= 0 \end{aligned} \quad (17)$$

$$\begin{aligned} G_{2n}(\xi, \xi) - G_{1n}(\xi, \xi) &= 0; \\ G_{2nx}(\xi, \xi) - G_{1nx}(\xi, \xi) &= 0; \\ G_{2nxx}(\xi, \xi) - G_{1nxx}(\xi, \xi) &= 1. \end{aligned} \quad (18)$$

Функция Грина задачи (14) имеет вид:

$$G_n(x, \xi) = \begin{cases} G_{1n}(x, \xi), & 0 \leq x < \xi, \\ G_{2n}(x, \xi), & \xi < x \leq p, \end{cases} \quad (19)$$

здесь

$$\begin{aligned} G_{1n}(x, \xi) &= \frac{1}{3\lambda_n^2 e^{\lambda_n p} \bar{\Delta}} \left(e^{-\lambda_n x} \left(2e^{\lambda_n \left(p - \frac{\xi}{2}\right)} \cos\left(\frac{\sqrt{3}}{2} \lambda_n \xi\right) - 2e^{\lambda_n \left(\xi - \frac{p}{2}\right)} \cos\left(\frac{\sqrt{3}}{2} \lambda_n p\right) \right) - \right. \\ &\quad \left. - 2e^{-\frac{\lambda_n}{2}(p-x)} \left(e^{\lambda_n \xi} + 2e^{-\frac{\lambda_n}{2} \xi} \cos\left(\frac{\sqrt{3}}{2} \lambda_n \xi\right) \right) \sin\left(\frac{\sqrt{3}}{2} \lambda_n (p-x) + \frac{\pi}{6}\right) + \right. \\ &\quad \left. + 2e^{\lambda_n \left(\frac{x}{2} - \frac{\xi}{2}\right)} \left(e^{\lambda_n p} + 2e^{-\frac{\lambda_n}{2} p} \cos\left(\frac{\sqrt{3}}{2} \lambda_n p\right) \right) \sin\left(\frac{\sqrt{3}}{2} \lambda_n (\xi-x) + \frac{\pi}{6}\right) \right), \\ G_{2n}(x, \xi) &= \frac{1}{3\lambda_n^2 e^{\lambda_n p} \bar{\Delta}} \left(e^{\lambda_n \xi} + 2e^{-\frac{\lambda_n}{2} \xi} \cos\left(\frac{\sqrt{3}}{2} \lambda_n \xi\right) \right) \cdot \\ &\quad \cdot \left(e^{\lambda_n (p-x)} - 2e^{-\frac{\lambda_n}{2}(p-x)} \sin\left(\frac{\sqrt{3}}{2} \lambda_n (p-x) + \frac{\pi}{6}\right) \right). \end{aligned}$$

Легко можно, убедиться, что функция, определенная формулой (19) обладает всеми свойствами, сформулированные при определении функции Грина.

Оценка $G_n(x, \xi)$ имеет вид

$$|G_n(x, \xi)| \leq \frac{K}{\lambda_n^2}, \quad \left| \frac{\partial G_n(x, \xi)}{\partial \xi} \right| \leq \frac{K}{\lambda_n}. \quad (20)$$

Интегрируя по частям второй интеграл в (16), имеем

$$V_n(x) = \lambda_n^3 \int_0^p G_n(x, \xi) f_n(\xi) d\xi + \int_0^p \left(\frac{\partial G_n(x, \xi)}{\partial \xi} a_1(\xi) + G_n(x, \xi) (a_1'(\xi) - a_2(\xi)) \right) V_n(\xi) d\xi. \quad (21)$$

Для удобства введём обозначения

$$V_{0n}(x) = \lambda_n^3 \int_0^p G_n(x, \xi) f_n(\xi) d\xi, \quad \bar{G}_n(x, \xi) = \frac{\partial G_n(x, \xi)}{\partial \xi} a_1(\xi) + G_n(x, \xi) (a_1'(\xi) - a_2(\xi)), \quad (22)$$

тогда (21) примет вид

$$V_n(x) = V_{0n}(x) + \int_0^p \bar{G}_n(x, \xi) V_n(\xi) d\xi. \quad (23)$$

Используя равенство (22), имеем оценку для $\bar{G}_n(x, \xi)$ в виде

$$|\bar{G}_n| \leq |a_1(\xi)| \left| \frac{\partial G_n(x, \xi)}{\partial \xi} \right| + |a_1'(\xi) - a_2(\xi)| |G_n| \leq \left(\frac{1}{\lambda_n} + \frac{1}{\lambda_n^2} \right) KQ.$$

Уравнение (23) является интегральным уравнением Фредгольма второго рода. Запишем решение (23) с помощью резольвенты в виде

$$V_n(x) = V_{0n}(x) + \int_0^p R_n(x, \xi) V_{0n}(\xi) d\xi$$

где

$$R_n(x, \xi) = \bar{G}_n(x, \xi) + \sum_{m=1}^{\infty} \bar{G}_{mn}(x, \xi), \quad (24)$$

здесь

$$\bar{G}_{mn}(x, \xi) = \int_0^p \bar{G}_n(x, s) \bar{G}_{(m-1)n}(s, \xi) ds, \quad m = 1, 2, \dots, \quad \bar{G}_{0n}(x, \xi) = \bar{G}_n(x, \xi).$$

Оценим решение (24). Из

$$R_n(x, \xi) = \bar{G}_{0n}(x, \xi) + \bar{G}_{1n}(x, \xi) + \bar{G}_{2n}(x, \xi) + \dots + \bar{G}_{mn}(x, \xi) + \dots$$

найдем оценку

$$|R_n(x, \xi)| \leq |\bar{G}_{0n}(x, \xi)| + |\bar{G}_{1n}(x, \xi)| + |\bar{G}_{2n}(x, \xi)| + \dots + |\bar{G}_{mn}(x, \xi)| + \dots$$

Для правой части этого неравенство составим мажорирующий ряд. Введя обозначение

$$J = \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_1^2} \right) KQ,$$

находим

$$|\bar{G}_{1n}(x, \xi)| \leq |\bar{G}_n(x, \xi)| \leq KQ \left(\frac{1}{\lambda_n} + \frac{1}{\lambda_n^2} \right) \leq \frac{1}{p} Jp,$$

$$|\bar{G}_{2n}(x, \xi)| \leq \int_0^p |\bar{G}_{1n}(x, s)| |\bar{G}_{1n}(s, \xi)| ds \leq \frac{1}{p} J^2 p^2,$$

$$|\bar{G}_{3n}(x, \xi)| \leq \int_0^p |\bar{G}_{1n}(x, s)| |\bar{G}_{2n}(s, \xi)| ds \leq \frac{1}{p} J^3 p^3,$$

.....

$$|\bar{G}_{mn}(x, \xi)| \leq \int_0^p |\bar{G}_{1n}(x, s)| |\bar{G}_{(m-1)n}(s, \xi)| ds \leq \frac{1}{p} J^m p^m, \quad m = 2, 3, \dots,$$

.....

Тогда мажорирующий ряд имеет вид

$$\frac{1}{p} \sum_{m=1}^{\infty} (Jp)^m$$

Если справедливо неравенство $Jp < 1$, то этот ряд является бесконечно убывающей геометрической прогрессией. Тогда должно выполняться соотношение

$$\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_1^2} \right) KQ < \frac{1}{p}$$

Отсюда имеем $Q < \frac{\lambda_1^2}{Kp(1 + \lambda_1)}$.

Существуют ряд значений чисел, которые удовлетворяют этому неравенству. Например, если $p = 1$, $q = 1$, то имеем $KQ < 2,38$.

В этом случае резольвента равномерно сходится, и её оценка имеет вид

$$|R(x, \xi)| \leq \frac{J}{1 - Jp} \leq M \quad (25)$$

Подставляя $G_n(x, \xi) = -\frac{1}{\lambda_n^3} G_{n\xi\xi\xi}(x, \xi)$ в $V_{0n}(x)$ и интегрируя, имеем

$$V_{0n}(x) = -f_n(x) + G_{n\xi\xi}(x, p) f_n(p) - \int_0^p G_{n\xi\xi}(x, \xi) f_n'(\xi) d\xi.$$

Учитывая оценки (15) и

$$|G_{2n\xi\xi}(x, 0)| \leq K, \quad |G_{1n\xi\xi}(x, p)| \leq K, \quad |G_{1n\xi\xi}(x, \xi)| \leq K,$$

получаем

$$|V_{0n}(x)| \leq \frac{M}{n^3} (|\Psi_{1n}| + |\Psi_{2n}| + |\Psi_{3n}| + |\bar{g}_n(x)| + |\bar{g}_n(0)| + |\bar{g}_n(p)| + |\bar{g}_n'(x)|), \quad (26)$$

Из (25) и (26) получим оценку

$$|V_n(x)| \leq |V_{0n}(x)| + \int_0^p |R(x, \xi)| |V_{0n}(\xi)| d\xi \leq \frac{M}{n^3} \left(\sum_{i=1}^3 |\Psi_{in}| + |\bar{g}_n(x)| + |\bar{g}_n(0)| + |\bar{g}_n(p)| + |\bar{g}_n'(x)| \right).$$

В силу (6) и (12) решение задачи А имеет вид

$$u(x, y) = \sum_{n=1}^{\infty} (V_n(x) + \rho_n(x)) \sin \left(\frac{\pi(2n-1)}{2q} y \right).$$

Проверим это решение на сходимость. Учитывая оценку

$$|\rho_n(x)| \leq \frac{M}{n^3} (|\Psi_{1n}| + |\Psi_{2n}| + |\Psi_{3n}|),$$

имеем

$$|u(x, y)| \leq \sum_{n=1}^{\infty} (|V_n(x)| + |\rho_n(x)|) \leq M \sum_{n=1}^{\infty} \frac{1}{n^3} \left(\sum_{i=1}^3 |\Psi_{in}| + |\bar{g}_n(x)| + |\bar{g}_n(0)| + |\bar{g}_n(p)| + |\bar{g}'_n(x)| \right).$$

Покажем сходимость $u_{xxx}(x, y)$. После некоторых вычислений, находим $V_n'''(x)$ в следующем виде.

$$V_n'''(x) = \lambda_n^3 f_n(x) - a_1 \left(V_{0n}'(x) + \int_0^p R_{nx}(x, \xi) V_{0n}(\xi) d\xi \right) - \\ - a_2 \left(V_{0n}(x) + \int_0^p R_n(x, \xi) V_{0n}(\xi) d\xi \right) - \lambda_n^3 \left(V_{0n}(x) + \int_0^p R_n(x, \xi) V_{0n}(\xi) d\xi \right).$$

Здесь

$$V_{0n}(x) = -f_n(x) + G_{n\xi\xi}(x, p) f_n(p) - \int_0^p G_{n\xi\xi}(x, \xi) f_n'(\xi) d\xi, \\ V_{0n}'(x) = -f_n'(x) + G_{n\xi\xi x}(x, p) f_n(p) - \int_0^p G_{n\xi\xi x}(x, \xi) f_n'(\xi) d\xi.$$

Используя оценки (26) и свойства функция Грина находим

$$|V_{0n}'(x)| \leq \frac{M}{n^3} (|\Psi_{1n}| + |\Psi_{2n}| + |\Psi_{3n}| + |\bar{g}_n(x)| + |\bar{g}_n(0)| + |\bar{g}_n(p)| + |\bar{g}'_n(x)|), \\ |R_{nx}(x, \xi)| \leq n^{\frac{2}{3}} M,$$

находим

$$|V_n'''(x)| \leq \frac{M^2}{n} \left(\sum_{i=1}^3 |\Psi_{in}| + |\bar{g}_n(x)| + |\bar{g}_n(0)| + |\bar{g}_n(p)| + |\bar{g}'_n(x)| \right).$$

Тогда имеем

$$|u_{xxx}(x, y)| \leq M^2 \sum_{n=1}^{\infty} \frac{1}{n} (|\Psi_{1n}| + |\Psi_{2n}| + |\Psi_{3n}| + |\bar{g}_n(x)| + |\bar{g}_n(0)| + |\bar{g}_n(p)| + |\bar{g}'_n(x)|).$$

Используя неравенства Коши-Буняковского и Бесселя:

$$|u_{xxx}(x, y)| \leq M^2 \left(\sqrt{\sum_{n=1}^{\infty} |\Psi_{1n}|^2} + \sqrt{\sum_{n=1}^{\infty} |\Psi_{2n}|^2} + \sqrt{\sum_{n=1}^{\infty} |\Psi_{3n}|^2} + \right. \\ \left. + \sqrt{\sum_{n=1}^{\infty} |\bar{g}_n(x)|^2} + \sqrt{\sum_{n=1}^{\infty} |\bar{g}_n(0)|^2} + \sqrt{\sum_{n=1}^{\infty} |\bar{g}_n(p)|^2} + \sqrt{\sum_{n=1}^{\infty} |\bar{g}'_n(x)|^2} \right) \sqrt{\sum_{n=1}^{\infty} \frac{1}{n^2}} \leq \\ \leq M \sqrt{\frac{\pi^2}{6}} \left(\|\psi_1'''\|_{L_2(0,q)} + \|\psi_2'''\|_{L_2(0,q)} + \|\psi_3'''\|_{L_2(0,q)} + \right. \\ \left. + \|g_y(x)\|_{L_2(0,q)} + \|g_y(0)\|_{L_2(0,q)} + \|g_y(p)\|_{L_2(0,q)} + \|g_{xy}(x)\|_{L_2(0,q)} \right) < \infty.$$

Здесь

$$\sum_{n=1}^{\infty} |\Psi_{in}|^2 \leq \|\psi_i'''\|_{L_2(0,q)}^2, \quad i = \overline{1, 3}, \quad \sum_{n=1}^{\infty} |\bar{g}_n(x)|^2 \leq \|g_y(x)\|_{L_2(0,q)}^2, \\ \sum_{n=1}^{\infty} |\bar{g}'_n(x)|^2 \leq \|g_{xy}(x)\|_{L_2(0,q)}^2, \quad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Учитывая неравенство

$$|u_{yy}(x, y)| \leq |u_{xxx}(x, y)| + |a_1(x)| |u_x(x, y)| + |a_2(x)| |u(x, y)|,$$

можно заключить что и u_{yy} тоже сходится.

Из решения задачи (14) получим решение задачи A в явном виде:

$$u(x, y) = \sum_{n=1}^{\infty} \left(\lambda_n^3 \int_0^p G_n(x, \xi) f_n(\xi) d\xi + \int_0^p \left(\bar{G}_n(x, \xi) + \sum_{m=1}^{\infty} \bar{G}_{mn}(x, \xi) \right) V_{0n}(\xi) d\xi + \right. \\ \left. + \rho_n(x) \right) \sin \left(\frac{\pi(2n-1)}{2q} y \right).$$

Таким образом, доказана теорема.

Литература

- [1]. Юлдашев Т.К. Обратная задача для одного интегро-дифференциального уравнения Фредгольма в частных производных третьего порядка // Вестн. Сам. гос. техн. ун-та. Сер. Физ.-мат. науки, - Самара, 2014. - № 1(34). - С.56-65.
- [2]. Рыжов О.С. Асимптотическая картина обтекания тел вращения со звуковым потоком вязкого и теплопроводящего газа // Прикл. Матем. и механ., - Москва, 1965. - Т. 29. Вып. 6. - С. 1004-1014.
- [3]. Диесперов В.Н. О функции Грина линейризованного вязкого трансзвукового уравнения // Журнал вычисл. матем. и матем. физики. - Москва, 1972. - Т. 12. - № 5. - С. 1265-1279.
- [4]. Block H. Sur les equations lineaires aux derives parielles a carateristiques multiples // Ark. Mat. Astron. Fus. Note 1, - 1912, 7(13), - pp. 1-34; Note 2, 1912, ibid. 7(21),- pp. 1-30; Note 3, 1912 - 1913, ibid. 8(23). - pp. 1-51.
- [5]. Del Vicchio E. Sulleequazioni $z_{xxx} - z_y + \varphi_1(x, y) = 0$, $z_{xxx} - z_{yy} + \varphi_2(x, y) = 0$ // Memorie R. Accad. Sci. Ser.2. - Torino, 1915, 66. - pp. 1-41.
- [6]. Cattabriga L. Potenziali di linea e di dominio per equazioni non paraboliche in due variabilia caratteristiche multiple // Rendiconti del seminario matimatico della univ. di Padova. - 1961, 31. - pp. 1-45.
- [7]. Джураев Т.Д., Апаков Ю.П. Об автомодельном решении одного уравнения третьего порядка с кратными характеристиками // Вестник Сам. гос. техн. ун-та. Сер. Физ.-мат. науки, - Самара, 2007. - № 2(15). - С.18-26.
- [8]. Джураев Т.Д., Апаков Ю.П. К теории уравнения третьего порядка с кратными характеристиками, содержащего вторую производную по времени // Украинский математический журнал. - Киев, 2010, том 62. № 1.- С. 40-51.
- [9]. Apakov Y.P., Rutkauskas S. On a boundary problem to third order PDE with multiple characteristics // Nonlinear Analysis: Modeling and Control. -Vilnius, 2011. - Vol. 16. -№ 3. - pp. 255-269.
- [10]. Апаков Ю.П. О решении краевой задачи для уравнения третьего порядка с кратными характеристиками // Украинский математический журнал. -Киев. 2012. Т.64. № 1. С. 1-11.
- [11]. Apakov Yu.P. On unique solvability of boundary-value problem for a viscous transonic equation // Lobachevski Journal of Mathematics. 2020.Vol 41, №. 6, -pp. 1754-1761. DOI: 10.1134/ S1995080220090036
- [12]. Апаков Ю.П., Жураев А.Х. О решении краевой задачи для уравнения третьего порядка с помощью функции Грина // Узбекский математический журнал. 2011, №3, - С.36-42.
- [13]. Yuldashev T.K., Apakov Y.P., Zhuraev A.Kh. Boundary value problem for third order partial integro-differential equation with a degenerate kernel. Lobachevski Journal of Mathematics.2021 Vol, 42, № 6, -pp. 1316-1326.
- [14]. Иргашев Б. Ю. Краевая задача для одного вырождающего уравнения высокого порядка с младшими членами //Bulletin of the Institute of Mathematics. – 2019. – №. 6. – С. 23-29.
- [15]. Сабитов К. Б. Об одной краевой задаче для уравнение смешанного типа третьего порядка в прямоугольной области//ДАН России. – Москва. 2009.-Т.427.-№5.-С.593-596.
- [16]. Sabitov K. B. Boundary Value Problem for a Third-Order Equation of Mixed Type in a Rectangular Domain, Differential Equations, 2013, 49 (2), 187–197.
- [17]. Sabitov K. B. Dirichlet Problem for a Third-Order Equation of Mixed Type in a Rectangular Domain, Differential Equations, 2011, 47 (5), 706–714.
- [18]. Сабитов К.Б., Удалова Г.Ю. Краевая задача для уравнения смешанного типа третьего порядка с условиями периодичности// Вестн. Сам. гос. техн. ун-та. Сер. Физ.-мат. науки, 2013, 3(32), -С. 29 - 45.
- [19]. Балкизов Ж.А., Кадзаков А.Х. О представлении решения краевой задачи для неоднородного уравнения третьего порядка с кратными характеристиками // Известия Кабардино - Балкарского научного центра РАН. - Нальчик, 2010. - № 4. - С. 64-69.
- [20]. Ashyralyev A., Ashyralyyev A., Abdalmohamme B. On the hyperbolic type differential equation with time involution, Bulletin of the Karaganda University. Mathematics series. 2023, 1 (109), pp. 38–47.
- [21]. Кожанов А.И., Дюжева А.В. Нелокальные задачи с интегральным условием для дифференциальных уравнений третьего порядка // Вестн. Сам. гос. техн. ун-та. Сер. Физ.- мат. науки, 2020. Т. 24, № 4. С. 607–620.

[22]. Apakov Yu.P., Sopuev A.A. Boundary value Problems for a mixed equation of Parabolic-Hyperbolic type of the third order // Lobachevski Journal of Mathematics. 2023. Vol 44, №. 12, -pp. 5155-5163. DOI: 10.1134/S1995080223120065

[23]. Apakov Y. P., Umarov R. A. On the third boundary problem for a nonhomogeneous third order equation with multiple characteristics, Uzbek Mathematical Journal, 2023, 67 (1), 4–14, DOI : 10.29229/uzmj.2023-1-1.

[24]. Apakov Y. P., Umarov R. A. Construction of the Solution of a Boundary-Value Problem for the Third-Order Equation with Lower Terms with the Help of the Green Function, Journal of Mathematical Sciences, 2023, 274 (6), 807–821, DOI : 10.1007/s10958-023-06644-2.

[25]. Apakov Y. P., Umarov R. A. Solution of a boundary value problem for a third-order inhomogeneous equation with multiple characteristics with the construction of the Green's function, Bulletin of the Karaganda University. Mathematics Series, 2024, 114 (2), 22–39, DOI : 10.31489/2024M2/22-39.

[26]. Апаков Ю. П., Умаров Р. А. О полноте собственных функций краевой задачи с несимметричными условиями для уравнения третьего порядка с кратными характеристиками. Международная научная конференция “Неклассические уравнения математической физики”, Ташкент. 24–26 октября 2024.

Kesik kubda ikkita aktiv bitta sust quvlovchilar va bir qochuvchining differensial o'yini

A.A.Adahamov, K.S.Kozimjonov

Annotatsiya. Ushbu ishda kesik kub qirralari bo'ylab harakatlanuvchi uch quvlovchi va bir qochuvchining quvlash differensial o'yinini o'rganilgan. Qochuvchining va ikki quvlovchining maksimal tezligi $\rho_{1,2} = 1$ va $\sigma = 1$ ga, uchinchi quvlovchining maksimal tezligi esa $1/2 \leq \rho_3 \leq 1$ ga teng. Agar biror quvlovchi va qochuvchining geometrik holatlari ustma-ust tushsa, qochuvchi tutilgan hisoblanadi. O'yinchilarning kesik kub qirralaridagi ixtiyoriy boshlang'ich holatlari uchun qochuvchini tutish mumkinligi isbotlangan. Bundan tashqari, quvlovchilar uchun tutish strategiyalari ham qurilgan.

Kalit so'zlar: Differensial o'yin, qochuvchi, quvlovchi, geometrik holat, kesik kub, joiz boshqaruvi, strategiya.

Аннотация. В данной работе исследуется дифференциальная игра преследования на усеченном кубе, где три преследователя и один убегающий перемещаются по ребрам усеченного куба. Максимальные скорости убегающего и двух преследователей равны $\rho_{1,2} = 1$ и $\sigma = 1$, а максимальная скорость третьего преследователя составляет $1/2 \leq \rho_3 \leq 1$. Считается, что убегающий пойман, если геометрические положения какого-либо преследователя и убегающего совпадают. Доказано, что убегающего можно поймать для любых начальных положений игроков на ребрах усеченного куба. Кроме того, построены стратегии преследования для преследователей.

Ключевые слова: Дифференциальная игра, убегающий, преследователь, геометрическое положение, усеченный куб, допустимое управление, стратегия.

Abstract. This work investigates a pursuit differential game on a truncated cube, where three pursuers and one evader move along the edges of the truncated cube. The maximum speeds of the evader and two pursuers are $\rho_{1,2} = 1$ and $\sigma = 1$, respectively, and the maximum speed of the third pursuer is $1/2 \leq \rho_3 \leq 1$. The evader is considered captured if the geometric positions of any pursuer and the evader coincide. It is proven that the evader can be captured for any initial positions of the players on the edges of the truncated cube. Furthermore, pursuit strategies for the pursuers are constructed.

Keywords: Differential game, evader, pursuer, geometric position, truncated cube, admissible control, strategy.

I. KIRISH

Bunday masalalarini o'rganish sal avvalroq boshlangan bo'lishiga qaramasdan, differensial o'yin tushunchasi amerikalik matematik Rufus Isaacs tomonidan fanga kiritilgan. Uning "Differensial o'yinlar" nomli kitobida u birinchi bo'lib differensial o'yinlar tushunchasini kiritgan.

Keyinchali boshqa matematiklar tomonidan differensial o'yinlarning bir nechta matematik asoslari ya'ni boshqacha qilib aytganda matematik formarizatsiyas yaratildi. Shulardan biri Lev Pontryagin tomonidan yaratildi. Differensial o'yinlarning bir nechta formarizatsiyasi bor. Bulardan asosan ikkitasini aytib o'tishimiz mumkin. Shulardan biri Lev Pontryagin tomonidan taklif qilingan formarizatsiya. Differensial o'yinlarda "Trayektoriya qanday qaraladi?" "Trayektoriya deb nimaga aytiladi?" "Strategiya nima?" kabi savollarga javob berish kerak bo'ladi. Lev Pontryagin mana shu savollarga javob berib, trayektoriyani aniqlab differensial o'yinlarga formarizatsiya yaratgan.

Lev Pontryagin formarizatsiga ko'ra, differensial o'yin ikkita nuqtai nazardan qaraladi: quvlovchi hamda qochuvchi. Bundan tashqari Lev Pontryagin nol cheksiz oraliqda qochib ketish masalari bo'yicha ham juda ulkan natijalar olgan. Lev Pontryagin quvlash masalari bo'yicha ikkita ulkan metod yaratdi.

Hozirgacha bu metodlar Pontryaginining birinchi metodi va Pontryaginining ikkinchi metodi deb yuritiladi.

Differensial o'yinlarning ikkinchi formarizatsiyasi Nikolay Krasovskiiga tegishli. Krasovskiy ilmiy maktabinin matematiklari tomonidan Isaacs metodi to'la asoslandi va u yerda zarur va yetarli shartlar darajasiga ko'tarildi. Chunki Isaacs tenglamasi umumanolganda yechimga ega bo'lmasligi yoki bitta yechimga ega bo'lishi yoki cheksiz ko'p yechimga ega bo'lishi mumkin.

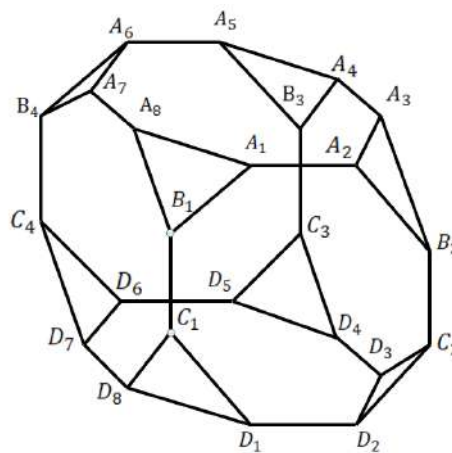
Endi jahon matematikasiga salmoqli hissa qo'shgan, differensial o'yinlar bo'yicha ulkan metodlar yaratgan matematiklarimi bilan tanishib chiqamiz. Shulardan biri No'monov Yunusovich Satimov. Satimov tomonidan

differentensial o'yinlar bo'yicha uchinchi metod yaratilgan. Bundan tashqari No'mon Yunusovich Satimov O'zbekistonga differentensial o'yinlar sohasini birinchi bo'lib olib kirgan matematik hisoblanadi. No'mon Yunusovich Satimov tomonidan nafaqat differentensial o'yinlar sohasida balki matematikaning boshqa sohalarida ham chuqur natijalar olingan.

Jahon matematikasiga salmoqli hissa qo'shgan, differentensial o'yinlar sohasida katta bir metod yaratgan navbatdagi matematigimiz O'zbekiston Fanlar Akademiyasi akademigi, fizika matematika fanlari doktori, O'zbekiston matematika jamiyatining raisi Abdulla

Azamovdir. Bu matematik nafaqat differentensial o'yinlar sohasida balki matematikaning barcha sohalarida ulkan natijalar olgan. Differentia o'yinlar sohasida katta bir metod yaratgan. Bu metod Azamovning to'rtinchi metodi deb yuritiladi.

Joriy ishda kesik kub K (1-rasm) qirralari bo'ylab quvlash differentensial o'yinini o'rganiladi. Barcha o'yinchilar kesik kubning qirralari bo'ylab harakat qilishadi. Ixtiyoriy boshlang'ich holatdan o'yinni tugatish uchun quvlovchilarning tezliklariga shart olingan. Bundan tashqari, quvlovchilar uchun tutish strategiyalari ham qurilgan.



1-rasm. K kesik kub.

II. MATERIALLAR VA USULLAR

Quyidagi

$$\dot{x}_i = u_i, \quad x_i(0) = x_{i0}, \quad i = 1, 2, 3$$

$$\dot{y} = v, \quad y(0) = y_0,$$

(1)

tenglamalar bilan beriladigan uchta x_1, x_2, x_3 quvlovchilar va bitta y qochuvchining differentensial o'yinini qaraymiz, bunda $x_{i0}, y_0 \in K$, $x_{i0} \neq y_0$, $i = 1, 2, 3$; u_i – i -quvlovchining, v esa qochuvchining boshqaruv parametri. Barcha o'yinchilar kesik kub K ning qirralari bo'ylab harakatlanadi, x_1, x_2, x_3 quvlovchilarning maksimal tezliklari, mos ravishda, ρ_1, ρ_2, ρ_3 larga, y qochuvchining maksimal tezligi esa σ ga teng, ya'ni $|u_i| \leq \rho_i$, $i = 1, 2, 3$, $|v| \leq \sigma$.

1-ta'rif. Agar ushbu

$$\dot{x}_i = u_i, \quad x_i(0) = x_{i0},$$

Koshi masalasining $x_i(\cdot)$ yechimi $x_i(t) \in K$, $t \geq 0$, shartni qanoatlantirsa, $u_i(\cdot)$, $u_i: [0, \infty) \rightarrow B(\rho_i)$, funksiya x_i , $i \in \{1, 2, 3\}$, quvlovchining joiz boshqaruvi deyiladi.

2-ta'rif. Agar ushbu

$$\dot{y} = v, \quad y(0) = y_0,$$

Koshi masalasining $y(\cdot)$ yechimi $y(t) \in K$, $t \geq 0$, shartni qanoatlantirsa, $v(\cdot)$, $v: [0, \infty) \rightarrow B(\sigma)$ funksiya y qochuvchining joiz boshqaruvi deyiladi.

Biz quvlash differentensial o'yinini qaraymiz. Bunday differentensial o'yinlarda quvlovchilar biror strategiyalarni qo'llaydi, qochuvchi esa ixtiyoriy joiz boshqaruvni qo'llaydi.

Endi quvlovchilarning strategiyalariga ta'rif beramiz.

3-ta'rif.

Agar $u_i =$

$U_i(t, x_1, x_2, x_3, y, v)$, $i = 1, 2, 3$, funksiyalar va qochuvchining ixtiyoriy $v = v(t)$ joiz boshqaruvi uchun (1) Koshi masalasi yagona $x_1(t), x_2(t), x_3(t), y(t) \in K$, $t \geq 0$, yechimga ega bo'lsa, $(t, x_1, x_2, x_3, y, v) \rightarrow U_i(t, x_1, x_2, x_3, y, v)$, $i = 1, 2$, funksiyalar mos ravishda x_i , $i = 1, 2, 3$, quvlovchilarning strategiyalari deyiladi.

4-ta’rif. Agar o‘yinchilarning ixtiyoriy boshlang‘ich holatlari uchun quvlovchilarning shunday strategiyalari topilib, biror $\tau \geq 0$ va $i \in \{1,2,3\}$ da $x_i(\tau) = y(\tau)$ bo‘lsa, u holda

III. NATIJALAR VA MUHOKAMA

Masala: (1) o‘yinda quvlovchilarga quvlashni tugatishni kafolatlaydigan strategiyalar qurishdan iborat.

Isbot. Quvlovchilar geometrik holatlarini X_1, X_2, X_3 qochuvchi vaziyatini esa Y bilan belgilaylik. Teoremani isbotlash uchun X_1, X_2 quvlovchining maksimal tezligi 1 ga, X_3 quvlovchining maksimal tezligi tezligi 1/2 teng bo‘lgan holni qarashimiz yetarli.

X_1, X_3 quvlovchilar A_1 uchda va X_2 quvlovchi esa A_5 uchda.

Endi quvlovchilarga qochuvchini tutish uchun strategiyalar quramiz.

Dastlab X_1, X_2 va X_3 quvlovchilar qochuvchini K kesik tetraedrning 2-rasmda qizil rangda ko‘rsatilgan $A_1A_2A_3 \dots A_8$ muntazam sakkizburcha va $S(B_1), S(B_2), S(B_3)$ va $S(B_4)$ uch yulduzlardan chiqarib yuborsin. Bu yerda $S(B_1)$ uch yulduz (B_1C_1, B_1A_8 va B_1A_1) qirralardan tuzilgan shuningdek $S(B_2), S(B_3)$ va $S(B_4)$ ham huddi shuningdek mos ravishda (B_2C_2, B_2A_2, B_2A_3), (B_3C_3, B_3A_4, B_3A_5) va (B_4C_4, B_4A_7, B_4A_6) qirralardan tuzilgan.

Agar qochuvchi t_1 vaqtda $A_4B_3A_5$ uchburchak va A_4A_3, A_5A_6, B_3C_3 qirralarda bo‘lsa, X_2 quvlovchi vaqtincha A_5 da harakatlanmay turadi. X_1 quvlovchi esa A_2, A_1 va A_8 nuqtalarni qo‘riqlaydi ya’ni qochuvchi A_3 dan A_2 ga harakatlansa X_1 A_1 dan A_2 ga $A_1X_1 = A_3Y$ tenglamaga mos ravishda harakat qiladi, ya’ni qochuvchi A_2 ga kelsa uni X_1 tutadi yoki qochuvchi A_2A_3 kesmada chiqib ketsa X_1 quvlovchi A_1 nuqtadagi boshlang‘ich holatiga keladi. Huddi shuningdek X_1 quvlovchi $A_1X_1 = B_2Y$ ga ham mos harakat qilib A_2 ni qo‘riqlaydi yoki $A_1X_1 = A_7Y$ tenglamaga mos ravishda A_8 ni qo‘riqlaydi. Yuqorida ta’kidlaganimizdek X_2 vaqtincha harakatlanmaydi X_3 esa $A_1A_2A_3 \dots A_8$ muntazam sakkiz burchakda soat strelkasiga qarshi harakat qilib A_4 nuqtga boradi va yana

quvlash tugadi yoki qochuvchi tutildi deb aytamiz.

Quvlovchilar mumkin qadar o‘yinni tez tugallashga harakat qilishadi. Qochuvchi esa iloji boricha o‘yinda quvlash tugashini kechiktirishga harakat qiladi.

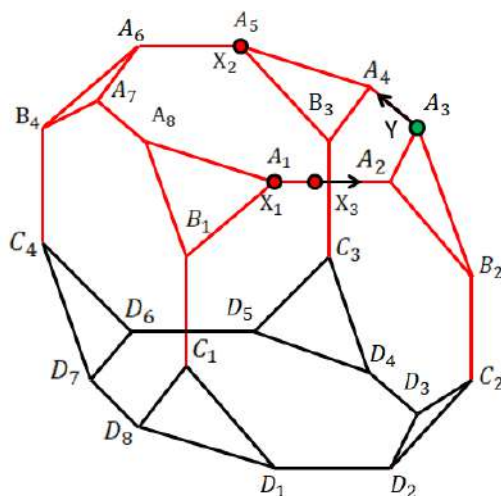
Bu masalaning yechimi quyidagi teoremadan iborat:

1-teorema. Agar $\rho_{1,2} = 1$ va $1/2 \leq \rho_3 \leq 1$ bo‘lsa, u holda kesik tetraedr K ning qirralari ustida qaralayotgan o‘yinda quvlashni tugatish mumkin.

saot strelkasiga qarshi ravishda $A_4A_5B_3$ da harakatlansin.

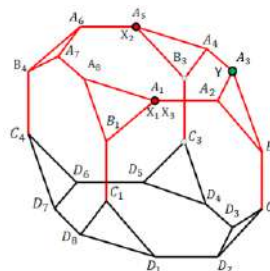
Agar X_3 quvlovchi A_4 nuqtaga kelganda qochuvchi A_4A_5 qirrada qolib ketsa X_2 yoki X_3 tomonidan tutiladi. Agar X_3 quvlovchi A_5 nuqtaga kelganda qochuvchi A_5A_6 qirrada harakatlanyotgan bolsa X_3 quvlovchi A_6 nuqta tomon harakatlanib qochuvchini tutadi yoki $A_6B_4A_7$ uchburchak ichiga kirishga majbur bo‘ladi.

ko‘rinib turibdiki qochuvchi B_4 yoki A_7



nuqtalardan biriga kelishga majbur boladi. Agar shu nuqtalardan biriga kelsa X_2 quvlovchiga strategiya quramiz. X_2 quvlovchi A_6 nuqtani qo‘riqlasin yani qochuvchi B_4 yoki A_7 nuqtadan A_6 nuqta tomon harakatlansa

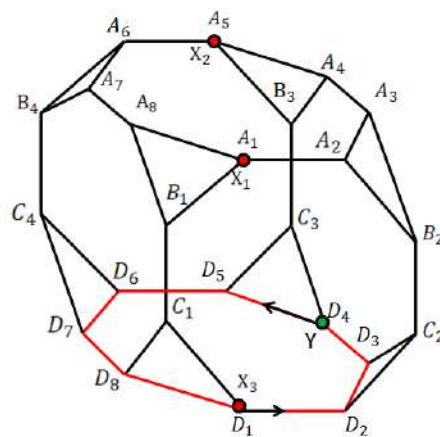
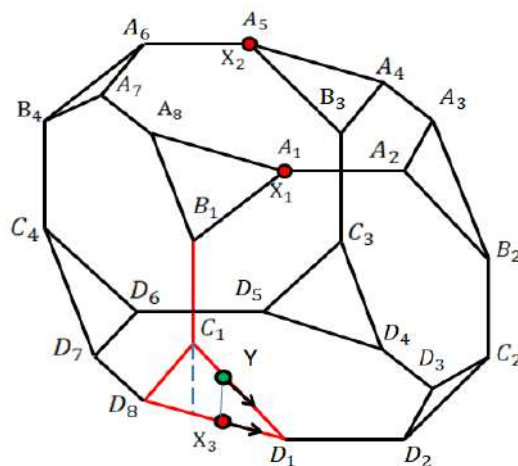
X_2 quvlovchi mos ravishda $A_5X_2 = B_4Y$ yoki $A_5X_2 = A_7Y$ tenglamalarga muvofiq ravishda harakat qiladi. Yani qochuvchi A_6 ga kelsa X_2 ham shu



nuqtaga bir vaqtda keladi va tutib oladi, agar qochuvchi B_4A_6 yoki A_7A_6 qirralardan chiqib ketsa X_2 quvlovchi A_5 nuqtada harakatlanmay turadi. Ana endi X_3 quvlovchi $A_6B_4A_7$ uchburchakda saot strelkasiga qarama-qarshi ravishda harakat qilsin. Agar X_3 quvlovchi A_6 nuqtaga kelganda qochuvchi A_6B_4 qirrada harakat qilayotgan bo'lsa X_3 quvlovchi B_4 nuqta harakat qilib qochuvchini tutadi yoki B_4C_4 yoki B_4A_7 qirralardan biriga yurishga harakat qiladi. Agar X_3 quvlovchi B_4 nuqtaga kelganda qochuvchi B_4A_7 qirrada harakatlansa X_3 quvlovchi A_7 nuqta tomon harakatlanib uni taqib qiladi va tutadi. Chunki A_6 va A_8 nuqtalarni mos ravishda yuqorida aytganimizdek X_2 va X_1 quvlovchilar qo'riqlaydi. Agar X_3 quvlovchi B_4 nuqtaga kelganda qochuvchi B_4C_4 qirrada harakatlansa X_3 uni ortidan harakat qilib uni tutadi yoki B_4C_4 qirradan chiqarib yuboradi. Aksincha X_3 quvlovchi A_5 ga kelganida qochuvchi A_5B_3 qirrada harakatlansa X_3 quvlovchi B_3 nuqta tomon harakatlanib qochuvchi C_3 yoki A_4 nuqta tomon harakatlanishiga majbur qiladi. Agar qochuvchi C_3 nuqta tomon harakatlansa X_3 quvlovchi C_3 nuqtada uni tutadi yuki B_3C_3 qirradan chiqarib yuboradi. Agar qochuvchi B_3 nuqtadan A_4 nuqta tomon harakatlansa X_3 quvlovchi B_3A_4 qirrada harakatlanib qochuvchini taqib qiladi va uni tutadi yoki B_3A_4 qirra tomon harakatlanishga majbur bo'ladi. Agar X_3 quvlovchi A_4 nuqtaga kelganda qochuvchi A_4A_3 qirrada bo'lsa X_3 quvlovchi A_3 tomon harakatlanib A_3 da tutadi yoki undan chiqarib yuboradi. Qochuvchi A_3 dan A_2 tomon harakatlansa X_1 quvlovchi $A_1X_1 = A_3Y$ ga mos ravishda harakatlanib qochuvchini tutadi yoki X_3 qochuvchi ortidan harakat qilib qochuvchini tutadi. Agar qochuvchi A_3 dan B_2 tomonga harakatlansa X_3 yana t

aqib qilib B_2 ga boradi. Qochuvchi B_2 dan A_2 tomon harakatlansa yana X_1 harakat qilib qochuvchini tutadi yoki X_3 tutadi. Shuning uchun qochuvchi B_2 dan C_2 tomon harakatlanadi va X_3 yana taqib qilib qochuvchini B_2C_2 qirrada tutadi yoki chiqarib yuboradi.

Huddi shuningdek Y qochuvchi t_1 vaqtda $A_4A_5B_3$ uchburchak hamda A_4A_3 , B_3C_3 , A_5A_6 qirralarda emas balki $A_1A_8B_1$



uchburchak va A_1A_2 , A_8A_7 , B_1C_1 qirralarda harakatlansin. Bu holatda X_1 quvlovchi vaqtincha harakatlanmay tursin, X_2 quvlovchi esa A_6 , A_5 , A_4 qirralarni qo'riqlaydi. Yuqorida keltirgan strategiyamizga o'xshash usul qo'llaymiz yani X_3 harakatlanib qochuvchini qizil chiziqdan chiqarib yuboradi.

Agar t_1 vaqtda qochuvchi $S(B_4)$ uch yulduz va A_6A_7 yoki $S(B_2)$ uch yulduz va A_2A_3 qirrada bo'lsa X_1 va X_2 mos ravishda yuqorida keltirilgan strategiyaga asosan A_8 , A_1 , A_2 va A_5 , A_4 , A_6 nuqtalarni qo'riqlaydi va X_3 quvlovchi qochuvchini qizil chiziqdan chiqarib yubora oladi.

t_2 vaqtdan boshlab X_1 va X_2 quvlovchilar mos ravishda B_1 va B_3 nuqtalarni qoriqlasin va mos ravishda C_2 va C_4 nuqtalarni poylasin. Bu vaqtda X_3 quvlovchi t_3 vaqtda D_1 nuqtaga kelsin.

$D_1D_2D_3...D_8$ sakkiz burchak boylab soat strelkasiga qarshi yo'nalishda $D_1 \rightarrow D_2 \rightarrow D_3 \rightarrow D_4 \rightarrow D_5 \rightarrow D_6 \rightarrow D_7 \rightarrow D_8 \rightarrow D_1$ kabi harakat qilsin. X_3 quvlovchi D_8D_1 qirra bo'ylab harakatlanayotganda qochuvchi $S(C_1)$

uchyulduzdan chiqib ketmasa, ravshanki biror t_4 vaqtda X_3 quvlovchi Y qochuvchini D_8D_1 qirradagi Y' proyeksiyasini tutib oladi ya'ni $X_3=Y'$ tenglikni saqlagan holda harakat qilsin. Bunda X_3 quvlovchi qochuvchini D_8 va D_1 nuqtaga keltirmaydi, chunki qochuvchi D_8 yoki D_1 nuqtaga kelsa X_3 uni tutib oladi. Agar D_8 yoki D_1 nuqtalarga kelmasa X_1 quvlovchi Y qochuvchi tomon harakatlanib uni tutadi.

Xuddi shuningdek t_4 vaqtda X_3 quvlovchi D_4D_5 qirra bo'ylab harakatlanayotgan vaqtda qochuvchi $S(C_3)$ uchyulduzdan chiqib ketmasa, ravshanki X_3 biror t_5 vaqtda qochuvchining D_4D_5 qirradagi proyeksiyasini tutadi va D_4 yoki D_5 nuqtalarga qochuvchini keltirmaydi. Agar kelsa X_3 quvlovchi bu nuqtalarda uni tutadi. Agra qochuvchi D_4 yoki D_5 nuqtalarga kelmasa X_2 quvlovchi qochuvchi tomon harakatlanib uni tutadi.

Agar t_4 vaqtda X_3 quvlovchi D_6D_7 qirralarda harakatlanayotganda qochuvchi D_6C_4 yoki D_7C_4 qirralarda qolib ketsa X_3 quvlovchi Y qochuvchini D_6D_7 qirradagi Y' C_4B_4 qirraga otishga harakat qilsa X_2 ham harakatini $A_7A_8D_4D_3$ tekislikka nisbatan qochuvchiga simmetrik ravishda harakat qiladi, ya'ni qochuvchi C_4B_4 qirradan B_4 ga kelsa X_2 ham A_5A_6 qirradan A_6 nuqtada keladi va huddi shuningdek uni A_7 nuqtaga keltirmaydi va B_4A_6 qirraning markaziga ham keltirmaydi chunki bu nuqtaga kelsa qochuvchi X_2 quvlovchi tomonidan tutiladi.

Aytaylik qochuvchi D_1 nuqtadan o'tsin. D_1 nuqtaga kelgan vaqt X_1 quvlovchi B_1C_1 , B_2C_2 , B_3C_3 va B_4C_4 qirralar ortalaridan o'tuvchi tekislikka nisbatan Y qochuvchiga simmetrik ravishda harakat qilsin va uni shu tekislikka keltirmaydi yoki shu tekislikda tutadi. Shuning uchun qochuvchi tekislikka yaqinlashmaydi. Huddi shu vaqtda ya'ni qochuvchi D_1 nuqtaga kelgan vaqt X_2 quvlovchi D_4 nuqta tomon harakatlanadi va $A_5 \rightarrow B_3 \rightarrow C_3 \rightarrow D_4$ kabi harakat bilan D_4 nuqtaga keladi. Mu'lumki A_5 nuqtadan D_4 nuqtagacha eng qisqa masofa uch birlik qirraga teng huddi shuningdek D_1 nuqtadan D_4 nuqtagacha eng qisqa masofa ham uch birlik qirraga teng. Shuning uchun qochuvchi eng qisqa yol bilan ham D_4 nuqtadan o'tib

proyeksiyasini tutadi va D_6 yoki D_7 nuqtalarga keltirmaydi aks holda uni tutadi. Agar qochuvchi C_4 nuqtadan o'tsa yoki shu nuqta orqali C_4B_4 qirra tomin harakat qilsa yuqorida aytqanimizdek X_2 quvlovchi C_4 nuqtani poylaydi va ravshanki qochuvchi Y yoki qochuvchi D_7 , D_6 yoki A_7 nuqtalardan hech biriga kelmasdan tursa X_1 quvlovchi qochuvchini taqib qilib uni tutadi. Huddi shuningdek X_3 quvlovchi D_2D_3 qirrada harakat qilayotganda qochuvchi D_2C_2 yoki C_2D_3 qirralarda qolib ketsa X_3 quvlovchi D_2D_3 qirradagi Y qochuvchini Y' soyasini tutadi va D_2 yoki D_3 nuqtalarga keltirmaydi va yuqoridagi kabi qochuvchi uchali quvlovchi tomonidan tutiladi.

Agar X_3 quvlovchi D_2D_3 , D_4D_5 , D_6D_7 , D_8D_1 qirralarda harakatlanayotganda qochuvchi mos ravishda $S(C_2)$, $S(C_3)$, $S(C_4)$ va $S(C_1)$ uchyulduzlarda qolib ketmasa, u holda biror t_5 vaqtda D_1 yoki D_5 nuqtalarning biridan o'tishga majbur boladi, aks holda X_3 quvlovchi qomonidan tutiladi.

ta olmaydi, agar D_4 ga kelsa X_2 quvlovchi tomonidan tutiladi. Qochuvchi D_1 nuqtaga kelgan vaqtda X_3 quvlovchi ham qochuvchi tomon harakat qilib uni taqib qilib D_2 nuqtaga keladi. Y qochuvchi D_2 nuqtaga kelganda yoki D_2D_3 orqali yuradi yoki $S(C_2)$ uchyulduzning D_2C_2 va C_2D_3 qirralar orqali yurishga majbur bo'ladi. Yuqorida aytqanimizdek X_3 quvlovchi D_2D_3 qirrada harakatlanayotganda qochuvchi $S(C_2)$ uchyulduzda qolib ketsa X_3 quvlovchi Y qochuvchini D_2D_3 qirradagi Y' proyeksiyasini malum bir vaqtda tutadi va D_2 va D_3 nuqtalarga keltirmaydi va o'z navbatida qochuvchi C_2B_2 qirraning ortasiga ham kelolmaydi chunki agar kelsa X_1 qulovchi tomonidan tutiladi. Agar quvlovchi hech qaysi nuqtalarga kelmasa X_2 quvlovchi qochuvchini taqib qilib uni tutadi. Agar qochuvchi $S(C_2)$ uch yulduz qirralarida emas balki D_2D_3 qirra bolsa X_3 quvlovchi qochuvchini D_3 nuqta tomon taqib qiladi va uni D_3D_4 qirraga otishiga majbur qiladi. Ravshanki D_4 nuqta X_2 quvlovchi tomonidan qo'riqlanayotganligi uchun qochuvchi D_3D_4 qirrada qamalib

qoladi va X_3 quvlovchi D_4 nuqta tomon harakat qilib qochuvchini tutadi.

Teorema isbotlandi.

XULOSA

Mazkur maqolada qaralgan Kesik kub – Arximed jismlari turkumiga kiradi. Ushbu ish Arximed jismlari qirralari ustidagi differensial o'yinlarga bag'ishlangan ishdur. Maqolada qaralgan masalada ikkita aktiv bitta sust quvlovchilar va bir qochuvchining ishtirokidagi differensial o'yin o'rganilgan. *Agar biror quvlovchi va qochuvchining geometrik holatlari ustma-ust tushsa, qochuvchi tutilgan hisoblanadi. O'yinchilarning kesik kub qirralaridagi ixtiyoriy boshlang'ich holatlari uchun qochuvchini tutish mumkinligi isbotlangan. Bundan tashqari, quvlovchilar uchun tutish strategiyalari ham qurilgan.*

Adabiyotlar

- [1] Azamov A.A., Ibragimov G.I., Ibaydullaev T., Alias I.A. (2021) Optimal number of pursuers in the differential games on the 1-skeleton of orthoplex. Symmetry. 10.3390/sym13112170.
- [2] Isaacs R. (1965). Differential games. John Wiley & Sons, New York, NY.
- [3] Fleming W.H. (1961). The convergence problem for differential games. Journal of Mathematical Analysis and Applications. 3, 102–116.
- [4] Satimov N.Y., Rikhsiev B.B. (2000). Methods of Solving of Evasion Problems in Mathematical Control Theory, Fan, Tashkent, Uzbekistan.
- [5] Chernous'ko F.L. (1976). A Problem of Evasion of Several Pursuers. Prikl. Mat. Mekh., 40(1): 14–24.
- [6] Pshenichnii B.N. (1976). Simple pursuit by several objects. Cybernetics and Systems Analysis, 12(3):145–146.
- [7] Kumkov S.S., Le Ménéec S., Patsko V.S. Zero-sum pursuit-evasion differential games with many objects: Survey of publications. Dynamic games and applications. 7: 609–633, Doi 10.1007/s13235-016-0209-z.
- [8] Ibragimov G.I., Salleh Y. (2012). Simple motion evasion differential game of many pursuers and one evader with integral constraints on control functions of players. Journal of Applied Mathematics, 2012, Article ID 748096, doi:10.1155/2012/748096.
- [9] Azamov A.A., Kuchkarov A.Sh., Holboyev A.G. (2019). The pursuit-evasion game on the 1-skeleton graph of the regular polyhedron. III. Mat. Teor. IgrPril., 11:4 (2019), 5–23.
- [10] Ramana M.V., and Kothari M. (2017) Pursuit strategy to capture high-speed evaders using multiple pursuers. Journal of Guidance, Control, and Dynamics. 40(1): 139-149, <https://doi.org/10.2514/1.G000584>.
- [11] Sun W., Tsiotras P. (2014) An optimal evader strategy in a two-pursuer one-evader problem, Proc. 53rd IEEE Conf. Decision and Control, Los Angeles, CA, 4266–4271.
- [12] Ibragimov G.I. (2023). Evasion differential game of one evader and many slow pursuers. Dynamic Games and Applications. <https://doi.org/10.1007/s13235-023-00501-2>.
- [13] Garcia E., Casbeer D.W., Pachter M. (2020). Optimal strategies for a class of multi-player reach-avoid differential games in 3D space. IEEE Rob. Autom. Lett., 5, 4257–4264, <https://doi.org/10.1109/LRA.2020.2994023>
- [14] M. Muhammadjonova, R.Ortiqova. Ikki burchakli beshburchak prizma qirralarida ikki quvlovchi va bir qochuvchining differensial o'yini. Scientific Bulletin of NamSU. 2025-4-son. pp.37-43.
- [15] Ibragimov G.I., Xasanjonova N.U. (2023) Kesik tetraedrda ikki quvlovchi va bir qochuvchining differensial o'yini. Scientific Bulletin of NamSU. № 7: 28–36.
- [16] Qo'shaqov X.Sh., Muhammadjonova M.A., Maxammadzokirov Sh.Sh. (2024). Differential game of three pursuers and one evader on an Elongated Square Gyrobicupola. Scientific Bulletin. Physical and Mathematical Research Vol. 6.
- [17] Qo'shaqov X.Sh., Maxammadzokirov Sh.Sh., Ortiqova R. (2024). Differential game of three pursuers and one evader on an elongated pentagonal pyramid. Scientific reports of Bukhara State University 2024/8 (113).

Kvadrat gumbaz qirralari bo'ylab ikki quvlovchi va bir qochuvchining tutish differensial o'yini

G. A. G'anijonova

Annotatsiya. Bu maqola Kvadrat gumbaz shaklida ikki quvlovchi yordamida bitta qochuvchining tutish differensial o'yinini o'zida aks ettiradi. O'yinda "asosiy quvlovchi" deb ataladigan quvlovchining maksimal tezligi $\rho_1 = 1$ bo'lib, bu qochuvchining maksimal tezligi bilan bir xil. Shuningdek, ikkinchi quvlovchining maksimal tezligi $\rho_2 = 1/2$. Quvlovchilardan birining holati qochuvchining holati bilan ustma-ust tushsa, quvlovchilar o'yinda g'alaba qozongan deb hisoblanadi. O'yinni yakunlash uchun strategiya qurdik va strategiyadan foydalangan holda tutish mumkinligi isbotladik.

Kalit so'zlar: Quvlovchi, qochuvchi, differensial o'yin, strategiya, kvadrat gumbaz.

Аннотация. В данной статье представлена дифференциальная игра по поимке одного беглеца с двумя преследователями в виде квадратного купола. В игре максимальная скорость преследователя, называемого «главным преследователем», равна $\rho_1 = 1$, что совпадает с максимальной скоростью беглеца. Также максимальная скорость второго преследователя равна $\rho_2 = 1/2$. Если позиция одного из преследователей совпадает с позицией беглеца, то преследователи считаются выигравшими игру. Мы построили стратегию для завершения игры и доказали, что с помощью стратегии можно поймать.

Ключевые слова: Преследователь, беглец, дифференциальная игра, стратегия, квадратный купол.

Abstract. This article presents a differential game of catching a single fugitive with two pursuers in the form of a square cupola. In the game, the maximum speed of the pursuer, called the "main pursuer", is $\rho_1 = 1$, which is the same as the maximum speed of the fugitive. Also, the maximum speed of the second pursuer is $\rho_2 = 1/2$. If the position of one of the pursuers coincides with the position of the fugitive, the pursuers are considered to have won the game. We have built a strategy to complete the game and proved that it is possible to catch using the strategy.

Keywords: Pursuer, fugitive, differential game, strategy, square cupola.

1. KIRISH

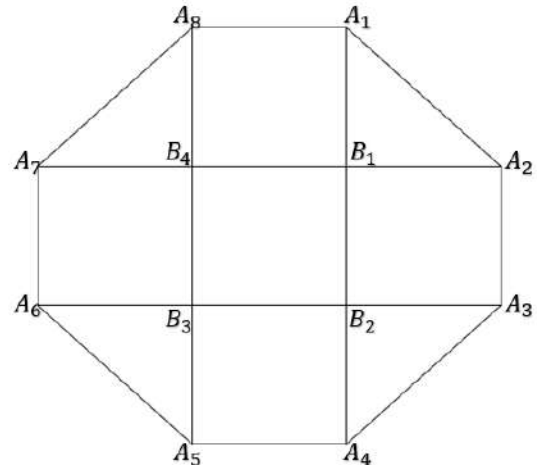
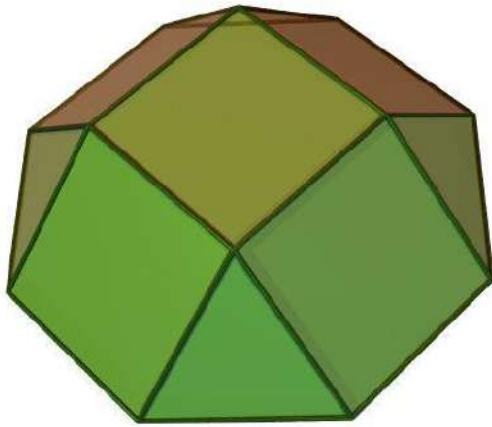
Differensial o'yin nazariyasi masalalari turli yo'nalishlarda o'rganilishda davom etib kelayotgan bo'lib, bu nazariya dastlab 1923-yili Shteyn Gauz maqolasidan boshlangan. So'ngra 1941-1945-yillarda ushbu nazariya uchun asos bo'ladigan turli harbiy sohadagi masalalar o'ranila boshlagan. 1950-yillarda Amerikalik matematik R.Isaacs Differensial o'yinlar nazariyasi bo'yicha birinchi bo'lib ma'lum bir sistemga ega bo'lgan o'z monografiyasini chop etadi va u "Differensial o'yinlar" deb ataladi. Ushbu monografiyada u ko'plab differensial o'yin muammolariga keltirilgan aniq model muammolari tahlilini taqdim etdi va ularning batafsil yechimlarini ko'rsatdi, shuningdek, bir qator yangi hayotiy muammolarini taklif qildi. 1960-yillarda Rossiyalik olimlar Pontryagin, Subbotin, Chikrii, Fleming, Petrosyan dinamik

o'yinlar va optimal boshqaruv metodlari va nazariyasini ishlab chiqqanlar.

O'zbek matematiklari orasida differensial o'yinlar sohasiga o'zining buyuk hissalarini qo'shib kelayotan olimlar Satimov N.Y., Azamov A.A., Ibragimov G.I., Ibaydullaev T.T., Kuchkarov A.Sh., Holboyev A.G. hisoblanib, bu sohada keng izlanishlar olib borilmoqda.

Differensial o'yinlarning bir nechta turlari mavjud. Xususan, ushbu ish ham ko'pyoq qirralarida ko'p quvlovchili differensial o'yinlar turiga mansub. Yana

shunday turdagi ko'plab masalalar Patsko v.b., Ibragimov va Salleh, Blagodetskix va Petrov, Azamov v.b., Ramana va Mangal, Sun va Tsiotras, Ibragimov v.b. Ibragimov, Garcia v.b. ishlarda keltirilgan.



Biz ushbu maqola orqali Kvadrat gumbaz shaklida quvlash masalasini o'rganamiz. Dastavval ushbu shakl bilan tanishib olamiz.

asosi kvadratdan hamda yon yoqlari 4 ta kvadrat va 4 ta muntazam uchburchakdan iborat ko'pyoq. Kvadrat gumbaz shakli 12 ta uch va 20 ta bir xil qirraga ega. Bu fazoviy shakl Arximed jismlarining bir turi hisoblanadi (1-rasm). 2-rasmda esa Kvadrat gumbazning grafi tasvirlangan. Biz qulaylik uchun Kvadrat gumbazni K harfi bilan belgilab olamiz.

Kvadrat gumbaz shunday fazoviy shaklki, u quyi asosi muntazam sakkizburchakdan, yuqori

2. MATERIALLAR VA USULLAR

Aytaylik tomoni bir-biriga teng bo'lgan ixtiyoriy $A_1 \dots A_8 B_1 \dots B_4$ Kvadrat gumbazda (2-rasm) $P = \{P_1; P_2\}$ quvlovchilar guruhi va bitta E qochuvchi harakatlanayotgan bo'lsin (2-rasm). Biz ularni umumiy o'yinchilar deb ataymiz. O'yinchilar doimo bir-birlarining harakatlari, holatlari 2-rasm. iradi deb olamiz. $P_i(t), i = 1, 2$ va $E(t)$ harakatlarini ravishda quvlovchilar va qochuvchining t momentdagi muntazam Tetraeksaedr shaklidagi vaziyati hisoblanadi. Jumladan, $t = 0$ momentdagi holat o'yinchilarning boshlang'ich holati deyiladi. Biz quvish o'yinini o'rganayotganimiz uchun quvlovchilarga o'yinni o'z foydalariga hal qilish uchun strategiyalar quramiz. Qochuvchi esa ixtiyoriy boshqaruvni qo'llab harakatlansin deb olamiz.

Qaralayotgan differensial o'yin quyidagicha berilgan:

$$\dot{P}_i(t) = u_i, P_i(0) = P_{i0}, i = 1, 2; \dot{E}(t) = v, E(0) = E_0$$

Bu yerda

$$P_i, E, u_i, v \in R^2, P_i(0) \neq E(0), \text{ va } |v| \leq \sigma = 1, |u_i| \leq \rho_i,$$

umumiylikka ziyon yetkazmagan holda, birinchi quvlovchi P_1 ning tezligini qochuvchining tezligiga, 1ga teng, deb olamiz ($\rho_1 = 1$), qolgan ikkinchi quvlovchining tezligi $1/2$ deb olamiz ($\rho_2 = 1/2$). P_{10}, P_{20} quvlovchilarning boshlang'ich holatlari va E_0 esa qochuvchining boshlang'ich holati. B(r) orqali radiusi r va markazi R^2 da bo'lgan doiralar to'plamini belgilaylik.

1-Ta'rif. Agar ushbu $\dot{P}_i(t) = u_i, P_i(0) = P_{i0}, i = 1, 2$ Koshi masalasining $P_i(\cdot)$ yechimi $P_i(t) \in K, t \geq 0$, shartni qanoatlantirsa $u_i(\cdot), u_i: [0, \infty) \rightarrow B(\rho_i)$ funksiya $P_i, i \in \{1, 2\}$, quvlovchilarning joiz boshqaruvi deyiladi.

2-Ta'rif. Agar ushbu $\dot{E}(t) = v, E(0) = E_0$, Koshi masalasining $E(\cdot)$ yechimi $E(t), t \geq 0$, shartni qanoatlantirsa, u holda $v(\cdot)$,

$v: [0, \infty) \rightarrow B(\sigma)$ funksiya E qochuvchining joiz boshqaruvi deyiladi.

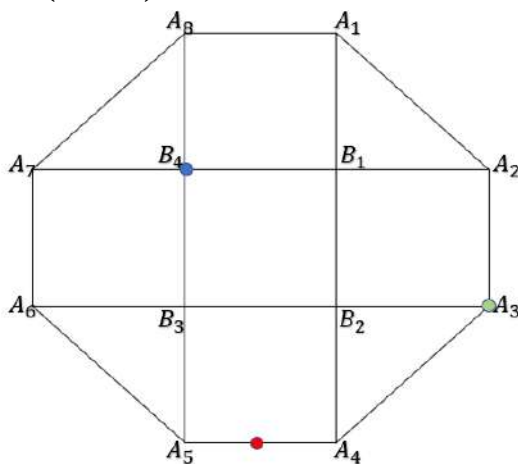
3-Ta'rif. Agar $u_i = U_i(t, x_1, x_2, y, v)$, $i = 1, 2$ funksiyalar va qochuvchining ixtiyoriy $v = v(t)$ joizboshqaruvi uchun berilgan Koshi masalasi yagona $P_1(t), P_2(t), E(t)$, $t \geq 0$, yechimga ega bo'lsa, $(t, P_1, P_2, E, v) \rightarrow U_i(t, P_1, P_2, E, v)$, $i = 1, 2$ funksiyalar mos ravishda P_i , $i = 1, 2$ quvlovchilarning strategiyalari deyiladi.

4-Ta'rif. Agar E qochuvchi muntazam tetraeksaedrning ixtiyoriy qirrasida joylashgan holda qanday strategiya qo'llamasin, $P = \{P_1, P_2\}$ guruhning shunday strategiyasi va unga binoan shunday $\tau > 0$ soni topilsaki, biror $i = 1, 2$ uchun $P_i(\tau) = E(\tau)$ tenglik o'rinli bo'lsa, u holda o'yinni quvlovchilar guruhi yutdi (ba'zan o'yin tugadi) deyiladi.

3. NATIJALAR VA MUHOKAMA

Teorema. Agar $\sigma = 1$ hamda $\rho_1 = 1$, va $\rho_2 = \frac{1}{2}$ bo'lsa, o'yinni P quvlovchilar guruh yutadi, ya'ni Kvadrat gumbaz qirralarida quvlovchilar guruhi qochuvchini tuta oladi.

Isbot. Umumiylikka ziyon yetkazmagan holda quvlovchilarni boshlang'ich holatda quyidagicha joylashtiramiz: $P_1(0) = B_4$; $P_2(0) = A_3$. Qochuvchi esa shaklning ixtiyoriy qirrasida joylashishi mumkin (3-rasm).



3-rasm.

Ko'k rangdagi doira tezligi qochuvchining tezligiga teng bo'lgan quvlovchining, yashil

rangdagi doira esa tezligi birinchi quvlovchining tezligining yarmiga teng bo'lgan quvlovchining holatini ifodalaydi. Qizil rangdagi doira esa qochuvchi.

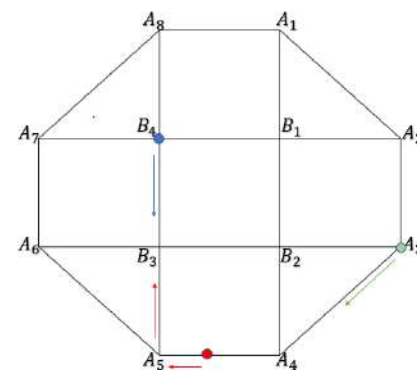
Dastlab qochuvchi $A_1A_2A_3A_4A_5A_6A_7A_8$ sakkizburchakda bo'lgan holni ko'raylik. Aytaylik, A_4A_5 qirrada bo'lsin. Tezligi $1/2$ ga teng bo'lgan A_3 uchda turgan P_2 quvlovchi harakatlanib quvlovchini yurishga majbur etsin. Qochuvchi harakatlanishi mumkin bo'lgan quyidagi hollarni ko'rib chiqamiz.

Birinchi holda qochuvchi A_5 uch tomon orqali A_6 uch tomon harakat qilsin. Qochuvchi harakatlanishni boshlashi bilan B_4 uchda turgan P_1 quvlovchi ham B_3 uch tomon harakatlana boshlaydi hamda bu nuqtada qochuvchi A_6 uchgayetib borguniga qadar harakatlanmay turadi. Qochuvchi A_6 uchga yetib borganda B_3 uchda turgan P_1 quvlovchi uning soyasini qochuvchi A_5 yoki A_6 nuqtada bo'lgan vaziyatda tutib oladi (4-rasm).

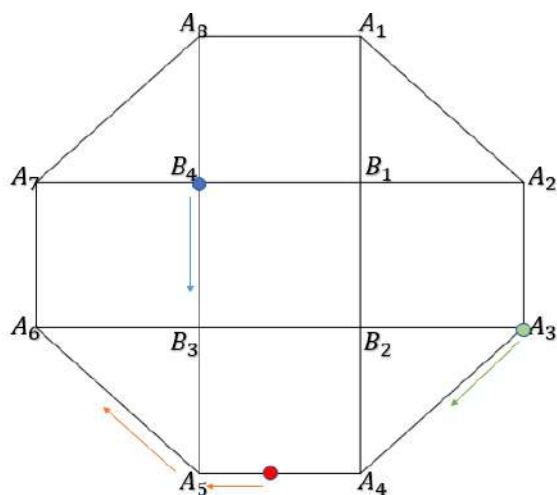
Soya deganda qochuvchi turgan joydan yon qirralariga parallel qilib o'tkazilgan to'g'ri chiziqning $B_1B_2B_3B_4$ kvadrat hamda $A_1A_2A_3A_4A_5A_6A_7A_8$ sakkizburchak bilan kesishish nuqtalari tushuniladi.

Soyasini tutib olgandan so'ng qochuvchining soyasi bilan harakatlana boshlaydi. Qochuvchi A_7 uchga yetib borganda u bilan simmetrik harakat qilib B_4 uchga yetib borgan P_1 quvlovchi $B_4A_7A_8$ uchburchakda A_8 uchdan B_4A_7 tomonga tushurilgan balandlikka nisbatan simmetrik harakat qiladi. Natijada qochuvchi A_8 uchda P_1 quvlovchi tumonidan tutiladi (5-rasm).

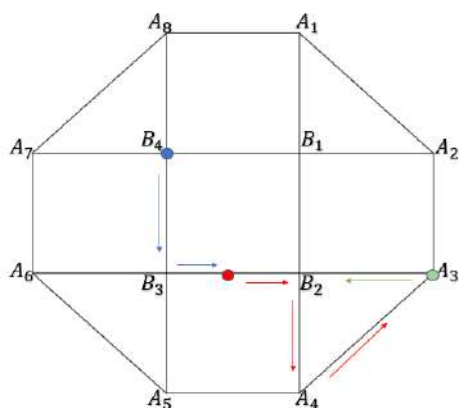
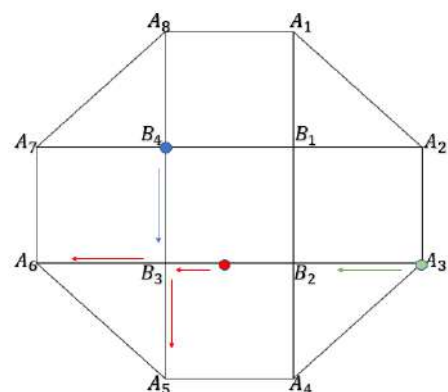
Ikkinchi holatda qochuvchi A_5 uch



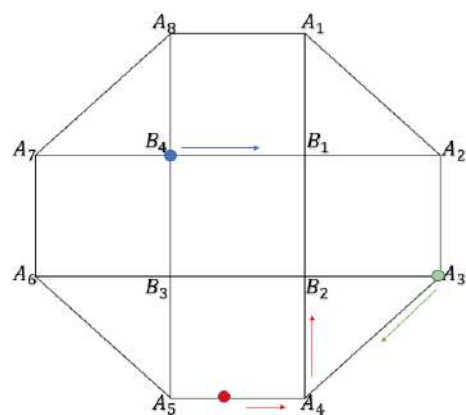
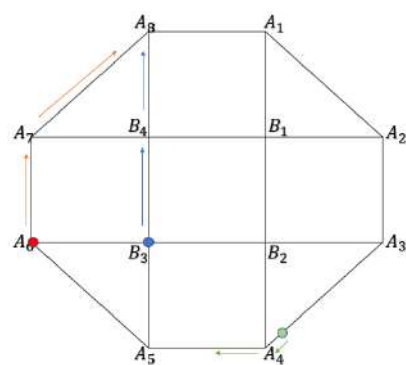
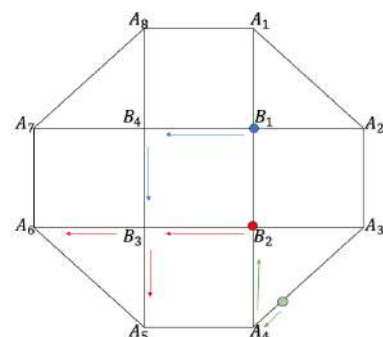
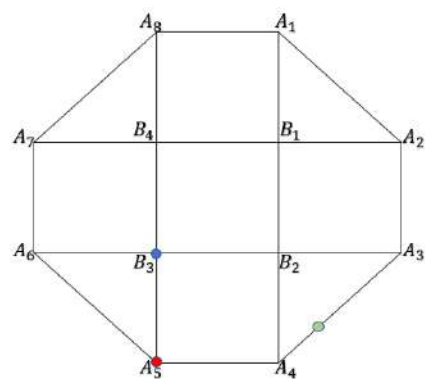
orqali B_3 uch tomon harakatlansin. . Qochuvchi



harakatlanishni boshlashi bilan B_4 uchda turgan P_1 quvlovchi ham B_3 uch tomon harakatlanish boshlaydi hamda bu uchga qochuvchidan avvalroq yetib keladi(6-rasm). Shu sababli qochuvchi A_5B_3 qirradan ortga qaytishga majbur bo'ladi. Natijada birinchi hodagidek



qochuvchining soyasi tutiladi. Demak bu holda ham qochuvchi P_1 quvlovchi tomonidan tutiladi(7-rasm)



Navbatdagi holda qochuvchi A_4 qirra orqali B_2 uch tomon harakatlansin. Qochuvchi A_3A_4 qirrada P_2 quvlovchi bo'lganligi sababli harakatlana olmaydi. Yuqoridagidek, qochuvchi harakatlanishni boshlashi bilanoq B_4 uchda turgan P_1 quvlovchi ham B_1 uch tomon harakatlana boshlaydi va yetib kelgandan so'ng harakatlanmay turadi(8-rasm). Qochuvchi P_1 quvlovchi B_1 uchga avvalroq yetib kelganligi sababli B_3 uch tomon harakatlana boshlaydi. Harakatlana boshlagandan so'ng P_1 quvlovchi ham B_4 uch tomon harakatlanadi. P_1 quvlovchi ham A_4 uchga yetib kelib, harakatini to'xtatmagan holda B_2 uch tomon harakatlanadi. Shu sababli qochuvchi ortga qayta olmaydi hamda B_3 uchdan A_5 yoki A_6 uch tomon harakatlanadi. Bu vaqtda B_4 uchga yetib kelgan P_2 quvlovchi B_3 tomon harakatlana boshlaydi. Natijada yana quvlovchining soyasini tutadi hamda birinchi holatdagidek qochuvchi bu holda ham P_1 quvlovchi tomonidan tutiladi(9-rasm). Endi to'rtinchi holni ko'rib chiqamiz. To'rtinchi holda quvlovchi Kvadrat gumbazning yuqori qismidagi $B_1B_2B_3B_4$ kvadratning ixtiyoriy biror qirrasida bo'lsin. Aytaylik B_2B_3 qirrada bo'lsin. A_3 uchda turgan P_2 quvlovchi B_2 uch tomon harakatlanib qochuvchini harakatlanishga majbur qiladi. Bunda ham quyidagi kichik hollarni ko'rib chiqamiz.

4.1-hol. Qochuvchi B_3 tomon harakatlansin. U holda B_3 uchga yetib borganidan so'ng B_4 , A_5 , A_6 uchlarga harakatlanishi mumkin. Ammo B_4 uchda P_1 quvlovchi turganligi sababli bu uch tomon harakatlana olmaydi. Qochuvchi B_3 uchdan A_5 yoki A_6 uch tomon harakatlanishni boshlash vaqtidan boshlab P_1 quvlovchi ham B_3 tomon harakatlana boshlaydi. Natijada yana soya tutiladi hamda birinchi holdagidek qochuvchi P_1 quvlovchi tomonidan tutiladi(10-rasm).

Qochuvchi B_2 uch orqali B_1 uch tomon harakatlansin. Bu holda P_1 quvlovchi qochuvchi harakatlanishni boshlagan vaqtdan boshlab B_1 uch tomon harakatlana boshlaydi. Natijada qochuvchi B_1B_2 qirrada tutiladi.

4.3-hol. Qochuvchi B_2 uch orqali A_4 uch tomon harakatlansin. P_1 quvlovchi qochuvchi A_4 tomon harakatlana boshlagan vaqtdan boshlab B_3 tomon harakatlanadi va qochuvchi A_4 uchga yetib borguniga qadar shu uchda qoladi. Agar qochuvchi A_5 uch tomon harakatlansa P_1 quvlovchi uning soyasini tutadi. Agar A_3 tomon harakatlansa, P_1 quvlovchi ham B_2 uch tomon harakatlanadi hamda soyasini tutadi. Agar qochuvchi harakatlanmay A_4 uchda turaversa, P_2 quvlovchi B_2 uch orqali borib qochuvchini A_3 yoki A_5 uclardan biri tomon harakatlanishga majbur qiladi hamda bu holda ham soya tutiladi. Natijda barcha hollarda birinchi hol singari qochuvchi tutiladi(11-rasm).

Oxiringi holatda ham yani qochuvchi Kvadrat gumbazning biror yon qirrasida bo'lgan holda ham P_2 quvlovchi yuqoridagi hollar kabi qochuvchini harakatlanishga majbur qilib qochuvchini gumbazning yuqori qismidagi kvadratga chiqaradi yoki pastki asosidagi sakkizburchakka tushiradi. Natijada bu hol uchun ham yuqoridagi 4 ta holatdagidek strategiya quriladi hamda quvlovchi tutiladi.

4. XULOSALAR

Ushbu maqolada Arximed jismlar turiga mansub Kvadrat gumbaz qirralarida 2 quvlovchi va bir qochuvchili differensial o'yin o'rganildi. Bunda bir quvlovchi va qochuvchining 11-rasm. tezliklari teng, umumiylikka ziyon yaratmagan holda bu tezlikni 1 deb oldik. Qolgan quvlovchining tezligi esa birinchi quvlovchi tezligining yarmiga teng deb oldik. O'yinda ikkinchi quvlovchi qochuvchini doimiy harakatlanishga majbur etuvchi quvlovchi bo'lib xizmat qilgan bo'lsa, birinchi quvlovchi qochuvchining harakatlanish maydonini kamaytirishga xizmat qildi. O'yinni qachon tugashligi $\min\{\rho_1; \rho_2\}$ qiymatga bog'liq. Kvadrat gumbaz qirralarida bir qochuvchini tutish uchun eng kamida 2 quvlovchi yetarli ekan. O'yin davomida quvlovchilarga o'yinni yakunlash uchun strategiyalar qurildi. [15] maqola ushbu ishni o'rganishimiz uchun asos bo'ldi.

Adabiyotlar

- [1] Azamov A., Ibaydullaev T. (2020). A pursuit-evasion differential game with slow pursuers on the edge graph of simplexes I. *Mathematical Game Theory and Applications*. 12(4): 7–23.
- [2] Azamov A.A., Ibragimov G.I., Ibaydullaev T., Alias I.A. (2021) Optimal number of pursuers in the differential games on the 1-skeleton of orthoplex. *Symmetry (Game Theoretical Symmetry Dynamic Processes)*. 10.3390/sym13112170.
- [3] Ibragimov G.I. (2023). Evasion differential game of one evader and many slow pursuers. *Dynamic Games and Applications*. <https://doi.org/10.1007/s13235-023-00501-2>.
- [4] Ibragimov G.I., Ferrara M., Ruziboev M., Pansera B.A. (2021). Linear evasion differential game of one evader and several pursuers with integral constraints. *International Journal of Game Theory*. doi.org/10.1007/s00182-021-00760-6
- [5] Ibragimov G.I., Salleh Y. (2012). Simple motion evasion differential game of many pursuers and one evader with integral constraints on control functions of players. *Journal of Applied Mathematics*, 2012, Article ID 748096, doi:10.1155/2012/748096.
- [6] Ibragimov G.I., Ferrara M., Kuchkarov A.Sh., and Pansera B.A. (2018). Simple motion evasion differential game of many pursuers and evaders with integral constraints. *Dynamic Games and Applications*. 8: 352–378. <https://doi.org/10.1007/s13235-017-0226-6>.
- [7] Isaacs R. (1965). *Differential games*. John Wiley & Sons, New York, NY, USA.
- [8] Krasovskii N.N., Subbotin A.I. (1988). *Game-Theoretical Control Problems*. Springer, New York.
- [9] Ibragimov G.I., Xasanjonova N.U. (2023) Kesik tetraedrda ikki quvlovchi va bir qochuvchining differensial o'yini. *Scientific Bulletin of NamSU*. № 7: 28–36.
- [10] Qo'shaqov X.Sh., Muhammadjonova M.A., Maxammadzokirov Sh.Sh. (2024). Differential game of three pursuers and one evader on an Elongated Square Gyrobicupola. *Scientific Bulletin. Physical and Mathematical Research* Vol. 6 Iss. 1.
- [11] Kumkov S.S., Le Ménec S., Patsko V.S. Zero-sum pursuit-evasion differential games with many objects: Survey of publications. *Dynamic games and applications*. 7: 609–633.

To'ldiruvchi fazoning o'lchami maksimal nilradikali filiform bo'lgan yechiluvchan Li algebrasining differentsiallashlari

Yuldashev Inomjon G'ulomboy o'g'li

Nukus davlat pedagogika instituti Matematika o'qitish metodikasi kafedrası assistent o'qituvchisi

i.yuldashev1990@mail.ru

ORCID: 0009-0004-4268-5420

UDK 517.98

Annotatsiya: So'nggi yillarda klassik tizimlarning assotsiativ bo'lmagan analoglari matematika va fizikaning ko'plab sohalarida qo'llanilib kelmoqda. Li va Leibniz algebralari kabi ba'zi noassotsiativ algebralar uchun lokal operatorlar, xususan lokal va 2-lokal differentsiallashlar tushunchalari ham keng o'rganila boshladi. Ushbu tushunchalarga aloqador asosiy muammolar har bir lokal (yoki 2-lokal) avtomorfizm va differentsiallash oddiy ma'noda avtomorfizm va differentsiallash bo'ladigan shartlarni topish, shuningdek, avtomorfizm bo'lmagan (mos ravishda differentsiallash) lokal va 2-lokal avtomorfizmga ega algebralar sinflarini aniqlashdan iboratdir. Hozirgi kunda jahonda differentsiallashlar nazariyasi bilan bir qatorda, operator algebralarda lokal va 2-lokal differentsiallashlar nazariyasi ham muhim hisoblanadi. So'nggi yigirma yil davomida Fon Neyman algebralari, C^* -algebralari va JB^* -uchliklarda lokal va 2-lokal differentsiallashlar nazariyasini o'rganish bo'yicha samarali natijalarga erishildi. Jumladan, Fon Neyman algebrasining qo'shma Banach M -bimoduli uchun har bir lokal differentsiallash oddiy ma'noda differentsiallash ekanligini isbotlagan. Bu natijani B.E.Djonson C^* -algebradagi differentsiallashlarni o'rganishga tatbiq qilib, C^* -algebraning ixtiyoriy uzluksiz lokal differentsiallashlari oddiy ma'noda differentsiallash ekanligi isbotlagan. Shuningdek, B.E.Djonson tomonidan C^* -algebradagi ixtiyoriy lokal differentsiallashlar uzluksiz bo'lishi isbotlangan. 1997 yilda P.Shemrl 2-lokal avtomorfizm va 2-lokal differentsiallash tushunchalarini kiritdi va u $B(H)$ -cheksiz o'lchamli separabel N Gilbert fazosida barcha chegaralangan chiziqli operatorlar algebrasida har bir 2-lokal avtomorfizm(mos ravishda, 2-lokal differentsiallash) avtomorfizm(mos ravishda, differentsiallash) bo'lishini isbotladi. Ta'kidlash joizki, P.Shemrl yuqoridagi teoremani H fazo chekli o'lchamli bo'lganda ham o'rinli bo'lishini ta'kidlagan. Keyinchalik Fon Neyman algebralari uchun 2-lokal differentsiallashlarni Sh.A.Ayupov, K.K.Kudaybergenov, F.N.Arzikulov, S.O.Kim, J.S.Kim, O.B.Nurjanov va A.K.Alauadinovlar o'rganishgan. Noassotsiativ algebralar jumladan, Leibniz algebralari uchun lokal va 2-lokal differentsiallashlarni o'rganish hozirgi kunning dolzarb muammolaridan biri hisoblanadi. Maqolada biz nilradikal filiform Li algebralari bo'lgan maksimal yechiladigan Li algebralari bo'yicha differentsiallarni o'rganamiz.

Kalit so'zlar: *Li algebra, differentsiallash, lokal differentsiallash, lokal avtomorfizm, 2-lokal differentsiallashlar, chiziqli operator.*

Abstract: In recent years, non-associative analogues of classical systems have found applications in many areas of mathematics and physics. For some non-associative algebras, such as Lie and Leibniz algebras, the concepts of local operators, in particular local and 2-local differentiations, have also begun to be widely studied. The main problems associated with these concepts are to find conditions under which every local (or 2-local) automorphism and differentiation are automorphisms and differentiations in the usual sense, and to determine classes of algebras that have local and 2-local automorphisms that are not automorphisms (derivations, respectively). At present, in addition to the theory of differentiation, the theory of local and 2-local differentiation in operator algebras is also of great importance in the world. Over the past twenty years, fruitful results have been achieved in the study of the theory of local and 2-local differentiations in von Neumann algebras, $-$ algebras, and $-$ triples. In particular, he proved that every local derivation for a joint Banach bimodule of a von Neumann algebra is a derivation in the usual sense. B.E. Johnson applied this result to the study of derivations in $-$ algebras and proved that arbitrary continuous local derivations of $-$ algebras are derivations in the usual sense. It was also proved by B.E. Johnson that arbitrary local derivations in $-$ algebras are continuous. In 1997, P. Schemrl introduced the notions of 2-local automorphism and 2-local derivation and proved that every 2-local automorphism (respectively, 2-local derivation) in the algebra of all bounded linear operators on an infinite-dimensional separable Hilbert space N is an automorphism (respectively, a derivation). It is worth noting that P. Schemrl emphasized that the above theorem is also valid in the case when the space is finite-dimensional. Later, 2-local differentiations for von Neumann algebras were studied by Sh. A. Ayupov, K. K. Kudaybergenov, F. N. Arzikulov, S. O. Kim, J. S. Kim, O. B. Nurzhanov and A. K. Alauadinov. The study of local and 2-local differentiations for non-associative algebras, including Leibniz algebras, is one of the topical problems of our time. In this article, we study differentials over maximally solvable Lie algebras, which are nilradical filiform Lie algebras.

Key words: *Lie algebra, derivation, local derivation, local automorphism, 2-local differentiations, linear operator.*

Kirish. A ba'zi bir algebra (associativ bo'lmagan). $D: A \rightarrow A$ chiziqli akslantirish barcha $x, y \in A$ elementlar uchun $D(xy) = D(x)y + xD(y)$ tengligini qanoatlantirsa, unda bu akslantirish differentsiyallash deyiladi. A chiziqli akslantirish Δ lokal differentsiyallash deyiladi, agar har bir $x \in A$ uchun D_x differentsiyallash bor bo'lib bunda A (x elementiga bog'liq) differentsiyallash ko'rinishi quyidagicha $\Delta(x) = D_x(x)$. Lokal avtomorfizmning ta'rifi ham xuddi shunday beriladi [1].

Adabiyotlar tahlili. Yuqoridagi tushunchalar R.V. Kadison ishlarida keltirilgan va o'rganilgan [1]. Shu bilan birgalikda D.R. Larson va A.R. Sourour ishlarida ham keltirilgan [2]. Yuqorida ta'kidlab o'tilgan ishlar avtomorfizmlar, C^* -algebralari va operator algebralarning differentsiyallashlariga yaqin bo'lgan akslantirishlarning tavsifiga bag'ishlangan bir qator ishlarga turtki bo'ldi. D.R. Larson va A.R. Sourourlar $A = B(X)$, tenglikni isbotlashdi, X Banach fazosidagi

barcha chegralangan chiziqli operatorlar algebrasi, u holda A ning har bir teskari lokal avtomorfizmi avtomorfizmdir. Shunday qilib $B(X)$ dagi avtomorfizmlar ularning lokal ta'siri yordamida to'liq aniqlanadi. Murakkab yarim oddiy chekli o'lchovli Leybnits algebrasidagi har bir lokal differentsiyallash differentsiyallash ekanligini isbotlangan [7].

Tadqiqot metodologiyasi.
Yechiluvchan Li algebralari

L Li algebrasi bo'lsin. Quyidagi markaziy qatorlarni qaraylik:

$$L^1 = L, \quad L^{k+1} = [L^k, L], \quad k \geq 1,$$

$$L^{[1]} = L, \quad L^{[s+1]} = [L^{[s]}, L^{[s]}], \quad s \geq 1.$$

L Li algebrasi nilpotent deyiladi (mos ravishda, yechiluvchan), agar shunday $p \in \mathbb{N}$ ko'rinishdagi $L^p = 0$ (mos ravishda, $L^{[p]} = 0$). Bunda k eng kichik butun son $L^k = 0$ tengligi L ning nilindeksi (yoki nilpotentlik klassi) deyiladi. [6]

Har qanday tabiiy ravishda baholangan filiform Li algebra quyidagi izomorf bo'lmagan algebralardan biriga izomorfdir:

$$n_{n,1}[e_i, e_1] = -[e_1, e_i] = e_{i+1}, \quad 2 \leq i \leq n-1,$$

$$Q_{2n}: \begin{cases} [e_i, e_1] = -[e_1, e_i] = e_{i+1}, & 2 \leq i \leq 2n-2, \\ [e_i, e_{2n+1-i}] = -[e_{2n+1-i}, e_i] = (-1)^i e_{2n}, & 2 \leq i \leq n. \end{cases}$$

Nilradikal tabiiy ravishda baholangan filiform Li algebra bo'lgan barcha hal qilinadigan

Li algebralari $n_{n,1}$ [4] da tasniflanadi. Nilradikal tabiiy ravishda baholangan filiform Lie algebra Q_{2n} bo'lgan yanada hal qilinadigan Li algebralari [11] da tasniflanadi. Nilradikal uchun izomorfik bo'lgan hal qilinadigan Li algebrasining o'lchami isbotlangan n -o'lchovli tabiiy ravishda baholangan filiform Li algebra dan katta emas $n+2$. Bu yerda biz bunday hal qilinadigan Li algebralari ro'yxatini beramiz. $S_{n,2}$ orqali $n_{n,1}$ nilradikali yechiluvchan Li algebralarni belgilaymiz. Xuddi shunday Q_{2n} nilradikalli Li algebralari uchun $\tau_{2n,2}$ belgilash olamiz.

$$s_{n,2} : \begin{cases} [e_i, e_1] = e_{i+1}, & 2 \leq i \leq n-1, \\ [e_1, x] = e_1, \\ [e_i, x] = (i-2)e_i, & 3 \leq i \leq n, \\ [e_i, y] = e_i, & 2 \leq i \leq n. \end{cases}$$

$$\tau_{2n,2} : \begin{cases} [e_i, e_1] = e_{i+1}, & 2 \leq i \leq 2n-2, \\ [e_i, e_{2n+1-i}] = (-1)^i e_{2n}, & 2 \leq i \leq n, \\ [e_i, x] = ie_i, & 1 \leq i \leq 2n-1, \\ [e_{2n}, x] = (2n+1)e_{2n}, \\ [e_i, y] = e_i, & 1 \leq i \leq 2n-1, \\ [e_{2n}, y] = -[y, e_{2n}] = 2e_{2n}. \end{cases}$$

Quyidagi teorema ushbu maqolaning asosiy natijasidir.

Olingan natijalar va muhokama

Teorema. $s_{n,2}$ va $\tau_{2n,2}$ algebralarning differentsiallari quyidagi shaklga ega:

$$Der(s_{2n,2}) : \begin{cases} d(e_1) = a_1 e_1 + \sum_{i=3}^n a_i e_i, \\ d(e_2) = b_2 e_2 - c_1 e_3, \\ d(e_i) = (b_2 + (i-2)a_1)e_i - c_1 e_{i+1}, & 3 \leq i \leq n-1, \\ d(e_n) = (b_2 + (n-2)a_1)e_n, \\ d(x) = c_1 e_1 + \sum_{i=3}^{n-1} (i-2)a_{i+1} e_i + c_n e_n, \\ d(y) = \sum_{i=2}^{n-1} a_{i+1} e_i + \frac{c_n}{(n-2)} e_n, \end{cases}$$

$$Der(\tau_{2n,2}) : \begin{cases} d(e_1) = (a_{2n+1} + a_{2n+2})e_1, \\ d(e_i) = (ia_{2n+1} + a_{2n+2})e_i + a_1 e_{i+1} + (-1)^i a_{2n+1-i} e_{2n}, & 2 \leq i \leq n \\ d(e_i) = (ia_{2n+1} + a_{2n+2})e_i + a_1 e_{i+1}, & n+1 \leq i \leq 2n-2 \\ d(e_{2n-1}) = (a_{2n+1} + a_{2n+2})e_{2n-1}, \\ d(e_{2n}) = ((2n+1)a_{2n+1} + 2a_{2n+2})e_{2n}, \\ d(y) = -2a_{2n} e_{2n}. \end{cases}$$

Isbot. Biz $s_{n,2}$ algebra uchun isbotni kiritamiz, $\tau_{2n,2}$ algebra uchun isbot o'xshash. d $s_{n,2}$ ning differentsiali bo'lsin. Biz o'rnatdik

$$\begin{aligned} d(e_1) &= \sum_{i=1}^n a_i e_i + a_{n+1}x + a_{n+2}y, & d(e_2) &= \sum_{i=1}^n b_i e_i + b_{n+1}x + b_{n+2}y, \\ d(x) &= \sum_{i=1}^n c_i e_i + c_{n+1}x + c_{n+2}y, & d(y) &= \sum_{i=1}^n \lambda_i e_i + \lambda_{n+1}x + \lambda_{n+2}y. \end{aligned}$$

Bundan tashqari bizda

$$\begin{aligned} d(e_1) &= d[e_1, x] = [d(e_1), x] + [e_1, d(x)] = \left[\sum_{i=1}^n a_i e_i + a_{n+1}x + a_{n+2}y, x \right] + \left[e_1, \sum_{i=1}^n c_i e_i + c_{n+1}x + c_{n+2}y \right] = \\ &= a_1 e_1 + \sum_{i=2}^n a_i (i-2) e_i - \sum_{i=2}^{n-1} c_i e_{i+1} + c_{n+1} e_1 = a_1 e_1 + a_3 e_3 + 2a_4 e_4 + 3a_5 e_5 + \\ &+ \dots + (n-2) a_n e_n - c_2 e_3 - c_3 e_4 - c_4 e_5 - \dots - c_{n-1} e_n + c_{n+1} e_1 = \\ &= (a_1 + c_{n+1}) e_1 + (a_3 - c_2) e_3 + (2a_4 - c_3) e_4 + \dots + ((n-2) a_n - c_{n-1}) e_n, \end{aligned}$$

Boshqa tomondan,

$$d(e_1) = a_1 e_1 + a_2 e_2 + K + a_n e_n + a_{n+1}x + a_{n+2}y.$$

Shuning uchun,

$$\begin{cases} a_2 = c_2 = a_{n+1} = a_{n+2} = c_{n+1} = 0 \\ c_3 = a_4 \\ c_4 = 2a_5 \\ \text{L L L} \\ c_{n-1} = (n-3)a_n \end{cases}$$

Quyidagi shartdan

$$\begin{aligned} 0 &= d[e_1, y] = [d(e_1), y] + [e_1, d(y)] = \left[a_1 e_1 + \sum_{i=3}^n a_i e_i, y \right] + \left[e_1, \sum_{i=1}^n \lambda_i e_i + \lambda_{n+1}x + \lambda_{n+2}y \right] = \\ &= \sum_{i=3}^n a_i e_i - \sum_{i=2}^{n-1} \lambda_i e_{i+1} + \lambda_{n+1} e_1 = a_3 e_3 + a_4 e_4 + K + a_n e_n - \lambda_{n+1} e_1 - \lambda_2 e_3 - \\ &- \lambda_4 e_5 - K - \lambda_{n-1} e = \lambda_{n+1} e_1 + (a_3 - \lambda_2) e_3 + (a_4 - \lambda_3) e_4 + K + (a_n - \lambda_{n-1}) e_n, \end{aligned}$$

Ushbu natijani olamiz

$$\begin{cases} \lambda_{n+1} = 0 \\ \lambda_2 = a_3 \\ \lambda_3 = a_4 \\ \text{L L L} \\ \lambda_{n-1} = a_n \end{cases}.$$

Quyidagini ko'rib chiqamiz

$$\begin{aligned} 0 = d[x, y] &= [d(x), y] + [x, d(y)] = \left[c_1 e_1 + \sum_{i=3}^{n-1} (i-2) a_{i+1} e_i + c_n e_n + c_{n+2} y, y \right] + \\ &+ \left[x, \lambda_1 e_1 + \sum_{i=2}^{n-1} a_{i+1} e_i + \lambda_n e_n + \lambda_{n+2} y \right] = \sum_{i=3}^{n-1} (i-2) a_{i+1} e_i + c_n e_n - \lambda_1 e_1 - \\ &- \sum_{i=2}^{n-1} (i-2) a_{i+1} e_i - (n-2) \lambda_n e_n = (c_n - (n-2) \lambda_n) e_n - \lambda_1 e_1. \end{aligned}$$

Bundan

$$\begin{cases} \lambda_n = \frac{c_n}{(n-2)}. \\ \lambda_1 = 0 \end{cases}$$

ega bo'lamiz.

Endi quyidagini ko'rib chiqamiz,

$$\begin{aligned} d(e_2) = d[e_2, y] &= \left[\sum_{i=1}^n b_i e_i + b_{n+1} x + b_{n+2} y, y \right] + \left[e_2, \sum_{i=2}^{n-1} a_{i+1} e_i + \frac{c_n}{(n-2)} e_n + \lambda_{n+2} y \right] = \\ &= \sum_{i=2}^m b_i e_i + \lambda_{n+2} e_2 = b_2 e_2 + b_3 e_3 + K + b_n e_n + \lambda_{n+2} e_2. \end{aligned}$$

boshqa tomondan,

$$d(e_2) = b_1 e_1 + b_2 e_2 + K + b_n e_n + b_{n+1} x + b_{n+2} y.$$

Shuning uchun,

$$b_1 = b_{n+1} = b_{n+2} = \lambda_{n+2} = 0.$$

Ushbu tenglikdan

$$0 = d[e_2, x] = \left[\sum_{i=2}^n b_i e_i, x \right] + \left[e_2, c_1 e_1 + \sum_{i=3}^{n-1} (i-2) a_{i+1} e_i + c_n e_n + c_{n+2} y \right] = \sum_{i=2}^n (i-2) b_i e_i + c_1 e_3 + c_{n+2} e_2$$

Quyidagiga ega bo'lamiz,

$$b_i = 0, i = \overline{4, n}, c_{n+2} = 0, b_3 = -c_1.$$

Xulosa. Demak, $S_{n,2}$ algebraning har qanday differensial qiymatga ega degan xulosaga kelamiz.

$$\begin{aligned}
 d(e_1) &= a_1 e_1 + \sum_{i=3}^n a_i e_i, \\
 d(e_2) &= b_2 e_2 - c_1 e_3, \\
 d(e_i) &= (b_2 + (i-2)a_1)e_i - c_1 e_{i+1}, \quad 3 \leq i \leq n-1, \\
 d(e_n) &= (b_2 + (n-2)a_1)e_n, \quad \dots \\
 d(x) &= c_1 e_1 + \sum_{i=3}^{n-1} (i-2)a_{i+1}e_i + c_n e_n, \\
 d(y) &= \sum_{i=2}^{n-1} a_{i+1}e_i + \frac{c_n}{(n-2)}e_n.
 \end{aligned}$$

Foydalanilgan adabiyotlar ro'yxati.

- [1]. R.V. Kadison, Local derivations, *J. Algebra*, 130 (1990) 494-509.
- [2]. D. R. Larson, A. R. Sourour, Local derivations and local automorphisms of $B(X)$, *Proc. Sympos. Pure Math.* **51** (1990) 187-194.
- [3]. B. Verbeke, Almost inner derivations of Lie algebras, PhD dissertation, KU Leuven Faculty of Science, 2020.
- [4]. Šnobl L., On the structure of maximal solvable extensions and of Levi extensions of nilpotent Lie algebras // *J. Phys. A, Math. Theor.* 43:17 (2010) Article ID 505202.
- [5]. Šnobl L., Winternitz P., A class of solvable Lie algebras and their Casimir invariants // *Journal of Physics A*, 38 (2005), 2687–2700.
- [6]. I.G'.Yuldashev. Local derivations on solvable Lie algebras with a filiform nilradical // "Qoraqalpog'istonda fan va ta'lim" 2021-yil 4-son (18)
- [7]. I.Kaygorodov, K.K.Kudaybergenov, I.G'.Yuldashev. Local derivations of semisimple Leibniz algebras // CIMPA School "Nonassociative Algebras and Its Applications", Madagascar. Communications in Mathematics 30 (2022), DOI: <https://doi.org/10.46298/cm.9274>.
- [8]. Ayupov Sh.A., Arzikulov F.N. Description of 2-local and local derivations on some Lie rings of skew-adjoint matrices // *Journal of Linear and Multilinear Algebra*. 68(4) (2020), p. 764-780.
- [9]. Ayupov Sh. A., Kudaybergenov K. K. 2-Local derivations and automorphisms on $B(H)$ // *Journal of Mathematical Analysis and Applications*. 395 (2012), p. 15–18.
- [10]. Ayupov Sh. A., Kudaybergenov K. K. 2-local derivations on von Neumann algebras // *Positivity*. 19 (2015), p. 445-455.
- [11]. J. M. Ancochea Bermúdez, R.Campoamor-Stursberg, L. Garca Vergnolle, *Solvable Lie algebras with naturally graded nilradicals and their invariants*, *J. Phys. A*, 39(6) (2006), 1339–1355.

Differential game of three pursuers and one evader on a rhombicuboctahedron

Abduraxmonova Robiyaxon Anvarjon qizi

Andijan davlat universiteti

E-mail: robixaxonabduraxmonova1311@gmail.com

Abstract. In this paper, we consider a differential game of three pursuers and one evader on a rhombicuboctahedron. In the game, two pursuers, which is called the "main pursuers" have maximum speeds of $\rho_1 = 1, \rho_2 = 1$. They are the same as the evader. However, the maximum speed of the other pursuer is just above 0, which means it is enough for it to move constantly. When the state of one of the pursuers coincides with the state of the evader, it is considered that the pursuers win the game. The strategy to complete the game is constructed, and it is proved that the pursuit is possible by using the strategy.

Key words: Pursuer, evader, differential game, strategy, rhombicuboctahedron.

Аннотация. В данной работе изучается дифференциальная игра трех преследователей и одного убегающего в вытянутом квадратном Ромбокубооктаэдре. Максимальные скорости двух преследователей, называемых в игре "главным преследователем", составляют $\rho_1 = 1, \rho_2 = 1$. Это то же самое, что и максимальная скорость убегающего. Однако у другого преследователя максимальная скорость немного выше 0, чего достаточно, чтобы он продолжал двигаться. Если позиция одного из преследователей совпадает с позицией убегающего, то преследователи считаются выигравшими игру. Была построена стратегия завершения игры, и было доказано, что с помощью этой стратегии можно поймать.

Ключевые слова: Преследователь, убегающий, дифференциальная игра, стратегия, Ромбокубооктаэдр.

Аннататсия. Ushbu maqolada biz rombikubo-oktaedr uch quvlovchi va bitta qochuvchi uchun differentsial o'yinni ko'rib chiqamiz. O'yinda "asosiy quvlovchilar" deb ataladigan ikkita quvlovchining maksimal tezliklari $\rho_1 = 1, \rho_2 = 1$. Bu qochuvchining maksimal tezligi bilan bir xil. Biroq, boshqa quvlovchining maksimal tezligi 0 dan biroz yuqoriroqdir, ya'ni u doimiy ravishda harakat qilishi uchun yetarli. Quvlovchilardan birining holati qochuvchining holati bilan ustma-ust tushsa, quvlovchilar o'yinda g'alaba qozongan deb hisoblanadi. O'yinni yakunlash uchun strategiya qurildi va strategiyadan foydalangan holda tutish mumkinligi isbotlandi.

Kalit so'zlar: Quvlovchi, qochuvchi, differentsial o'yin, strategiya, rombikubo-oktaedr.

1. Kirish

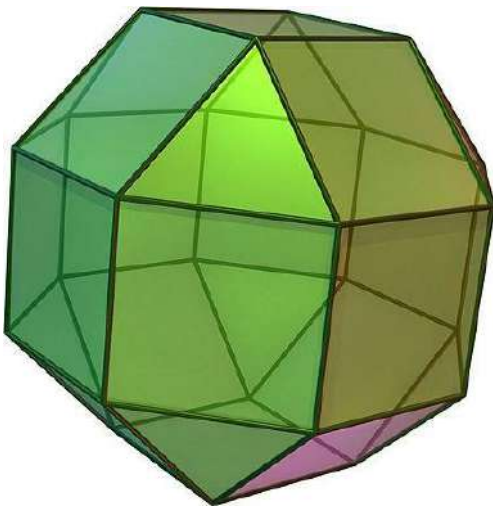
O'yinlar nazariyasi matematikaning noaniqlik mavjud bo'lgan vaziyatlarda optimal qaror qabul qilish masalalarini o'rganadigan bo'limdir. Ya'ni o'z manfaatlarini ko'zlovchi ikki tomonlarning kurashuvidir. Ularning o'yin davomida qabul qilgan qarorlari strategiya deyiladi. U ma'lum maqsadga yo'naltirilgan bo'ladi.

Yaqin yillarda o'zbek matematik olimlar tomonidan differentsial o'yinlarda amalga oshirilayotgan ishlar sohani rivojlantirib kelmoqda. Jumladan, Satimov N.Y. [2], Azamov A.A. [1], Ibragimov G.I. [3], [4], [5], Ibaydullaev T.T. [6], kabi matematik olimlar sohaning rivojiga ulkan hissa qo'shib kelishmoqda.

Arximed jismlari birinchi bo'lib Arximed tomonidan o'rganilgan. Biroq uning bu sohadagi asarlari bizgacha yo'qolgan. 1619-yilga kelib nemis matematigi Yoxannes Kepler

ularni qayta tikladi va ularga yunoncha nom berdi. rombikubo-oktaedr ham Arximed jismlarining biri hisoblanadi.

Biz ushbu maqola orqali rombikubo-oktaedrda quvlash masalasini o'rganamiz. rombikubo-oktaedr shunday fazoviy shaklki, 8 ta muntazam uchburchak va 18 ta kvadratlil yoqdan iborat yo'pyoq. cho'zilgan kvadrat girobikupolada 24 ta uch bo'lib, ularning har birida 1ta uchburchak va 3ta kvadrat kesishadi va 24 ta bir xil qirraga ega. Bu fazoviy shakl Arximed jismlarining bir turi hisoblanadi (1-rasm). 2-rasmda esa rombikubo-oktaedrning grafi tasvirlangan. [7] maqola aynan Arximed jismlaridagi differentsial o'yinlarga bag'ishlangan birinchi ishdir. [8] ish aynan shu shaklda birinchi bo'lib o'ynalgan differentsial o'yin hisoblanadi. Ushbu maqola esa [8] ishdan farqli bo'lgan strategiya orqali tuzilgan differentsial o'yinni o'z ichiga oladi



1-rasm. Rhombicuboctahedron

2. Masalaning qo'yilishi

Aytaylik tomoni bir birlik bo'lgan ixtiyoriy

$A_1, \dots, A_4, B_1, \dots, B_8, C_1, \dots, C_8, D_1, \dots, D_4$ Rombikubo-oktaedrning grafida (2-rasm) $P = \{P_1; P_2; P_3\}$ quvlovchilar guruhi va bitta E qochuvchi harakatlanayotgan bo'lsin (2-rasm). Biz ularni umumiy o'yinchilar deb ataymiz. O'yinchilar doimo bir-birlarining harakatlarini, holatlarini ko'rib, his qilib turadi deb olamiz. $P_i(t), i = 1, 2, 3$ va $E(t)$ lar mos ravishda quvlovchilar va qochuvchining t momentdagi Rombikubo-oktaedrdagi vaziyati hisoblanadi. Jumladan, $t = 0$ momentdagi holat o'yinchilarning boshlang'ich holati deyiladi. Biz quvish o'yinini o'rganayotganimiz uchun quvlovchilarga o'yinni o'z foydalariga hal qilish uchun strategiyalar quramiz. Qochuvchi esa ixtiyoriy boshqaruvni qo'llab harakatlansin deb olamiz.

Qaralayotgan differensial o'yin quyidagicha berilgan:

$$\dot{P}_i(t) = u_i,$$

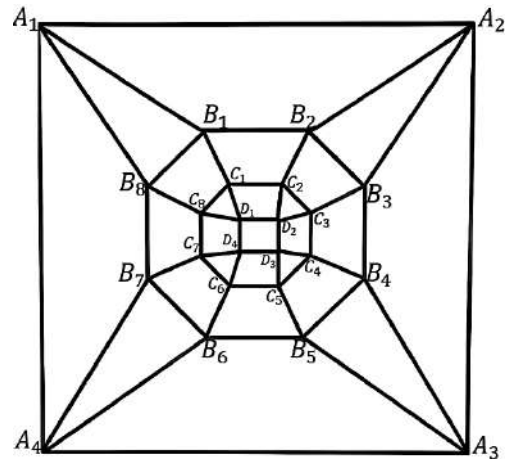
$$P_i(0) = P_{i0}, \quad i = 1, 2, 3$$

$$\dot{E}(t) = v, \quad E(0) = E_0$$

bu yerda

$P_i, E, u_i, v \in R^2, P_i(0) \neq E(0),$ va $|v| \leq \sigma = 1, |u_i| \leq \rho_i.$

umumiylikka ziyon yetkazmagan holda, birinchi va ikkinchi quvlovchilarning tezliklarini qochuvchining tezligiga, 1 ga teng, deb olamiz ($\rho_1 = 1, \rho_2 = 1$), uchinchi



2-rasm. Rhombicuboctahedron grafi

quvlovchining tezligi nomigagina mavjud bo'lsa, kifoya, ya'ni 0 dan biroz katta bo'lsa, yetadi ($0 < \rho_3 < 1$). P_{10}, P_{20}, P_{30} quvlovchilarning boshlang'ich holatlari va E_0 esa qochuvchining boshlang'ich holati. $B(r)$ orqali radiusi r va markazi R^2 da bo'lgan doiralar to'plamini belgilaylik.

Biz qulaylik uchun Rombikubo-oktaedrning qirralarini R bilan belgilaymiz.

1-Ta'rif. Agar ushbu

$$\dot{P}_i(t) = u_i, P_i(0) = P_{i0}, i = 1, 2, 3$$

Koshi masalasining $P_i(\cdot)$ yechimi $P_i(t), t \geq 0$, shartni qanoatlantirsa, u holda $u_i(\cdot), u_i: [0, \infty) \rightarrow B(\rho_i),$ funksiya $P_i, i \in \{1, 2, 3\},$ quvlovchilarning joiz boshqaruvi deyiladi.

2-Ta'rif. Agar ushbu

$$\dot{E}(t) = v, E(0) = E_0,$$

Koshi masalasining $E(\cdot)$ yechimi $E(t), t \geq 0$, shartni qanoatlantirsa, u holda $v(\cdot), v: [0, \infty) \rightarrow B(\sigma)$ funksiya E qochuvchining joiz boshqaruvi deyiladi.

Biz quvlash differensial o'yinini qaraymiz. Bunday differensial o'yinlarda quvlovchilar biror strategiyalarni qo'llaydi, qochuvchi esa ixtiyoriy joiz boshqaruvni qo'llaydi.

Endi quvlovchilarning strategiyalariga ta'rif beramiz.

3-Ta'rif. Agar

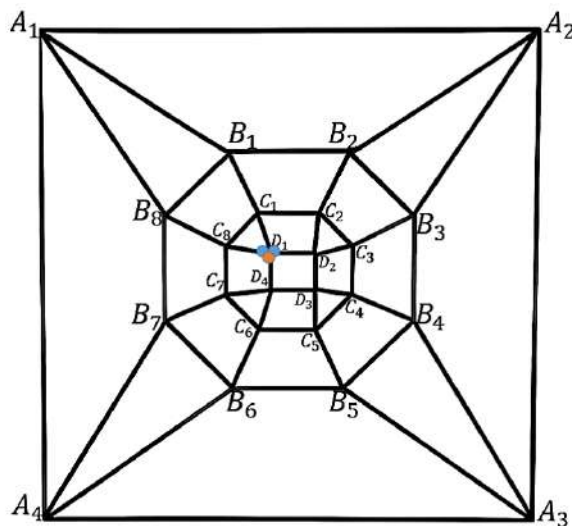
$$u_i = U_i(t, P_1, P_2, P_3, E, v), \quad i = 1, 2, 3$$

funksiyalar va qochuvchining ixtiyoriy $v = v(t)$ joiz boshqaruvi uchun berilgan Koshi masalasi yagona $P_1(t), P_2(t), P_3(t), E(t), t \geq 0,$ yechimga ega bo'lsa,

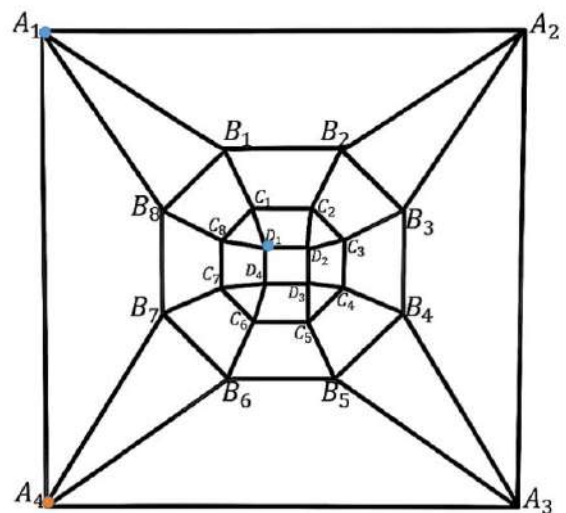
$(t, P_1, P_2, P_3, E, v) \rightarrow U_i(t, P_1, P_2, P_3, E, v)$,
 $i = 1, 2, 3$ funksiyalar mos ravishda P_i , $i = 1, 2, 3$ quvlovchilarning strategiyalari deyiladi.

4-Ta'rif. Agar E qochuvchi Rombikubo-oktaedrning ixtiyoriy qirrasida joylashgan holda qanday strategiya qo'llamasin, $P = \{P_1; P_2; P_3\}$ guruhning shunday strategiyasi va unga binoan shunday $\tau > 0$ soni topilsaki, biror $i = 1, 2, 3$ uchun $P_i(\tau) = E(\tau)$ tenglik o'rinli bo'lsa, u holda o'yinni quvlovchilar guruhi yutdi (ba'zan o'yin tugadi) deyiladi.

Quyidagi teorema ushbu maqolaning asosiy natijasi hisoblanadi.



a) 3-rasm.



b)

Ko'k rangdagi doiralar tezligi qochuvchining tezligiga teng bo'lgan quvlovchlarni, qizg'ish rangdagi doira esa tezligi nomigagina mavjud bo'lgan quvlovchini ifodalaydi.

Dastlab P_1 quvlovchi o'z o'rnida, D_1 uchda qoladi. P_2 quvlovchi 1 tezlikda A_1 uch tomon va P_3 quvlovchi 0 dan biroz katta bo'lgan tezlikda A_4 tomon harakatlanadi. Bu holatda quvlovchilar qochuvchining qanday harakat qilishiga e'tibor qaratishmaydi va $t_1 = 3 + \frac{4}{p_3}$ vaqtga kelib o'yinchilarning holati quyidagicha:

3. Asosiy natija

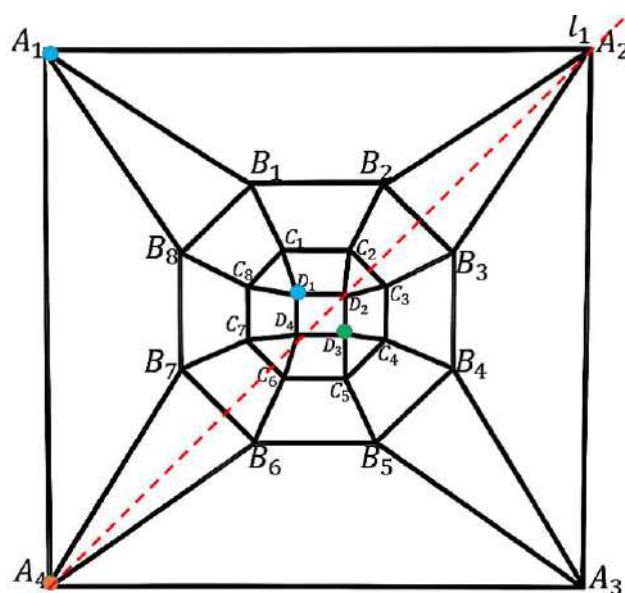
Teorema. Agar $\sigma = 1$ hamda $\rho_1 = 1$, $\rho_2 = 1$ va $\rho_3 > 0$ bo'lsa, o'yinni P quvlovchilar guruhi yutadi, ya'ni Rombikubo-oktaedrning qirralarida quvlovchilar guruhi qochuvchini tuta oladi.

Isbot. Umumiylikka ziyon yetkazmagan holda, quvlovchilarni boshlang'ich holatda D_1 uchda turgan bo'lishsin deb olamiz, ya'ni $P_i(0) = D_1$, $i = 1, 2, 3$. Qochuvchi esa Rombikubo-oktaedrning istalgan nuqtasida joylashgan bo'lsin (3a-rasm).

$P_1(t_1) = D_1, P_2(t_1) = A_1, P_3(t_1) = A_4$.
 Qochuvchi esa Rombikubo-oktaedrning ixtiyoriy nuqtasida joylashgan (3b-rasm).

t_1 vaqtdan boshlab avvalo uchta "Umumiy holat" deb ataluvchi holatlarni ko'rib chiqamiz:

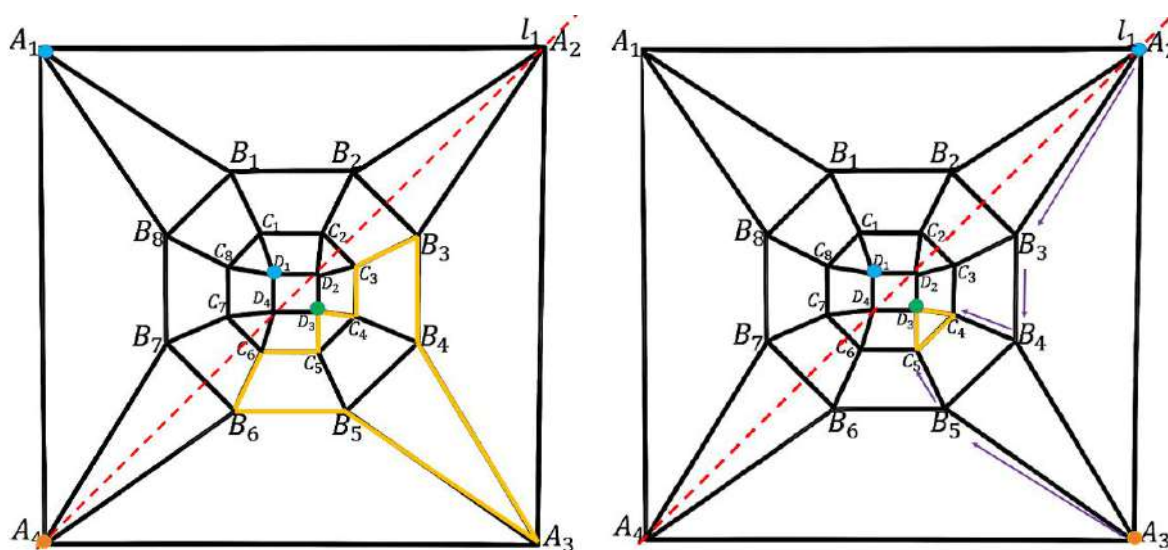
1. Birinchi holatda E qochuvchi t_1 vaqtda D_3 nuqtada bo'lsin. Ya'ni bu holatda $P_1(t_1)$ (birinchi quvlovchining holati) va $E(t_1)$ (qochuvchining holati, yashil doira bilan ifodalangan) nuqtalar l_1 to'g'ri chiziqqa nisbatan simmetrik joylashgan bo'lsin. (4-rasm).



4-rasm. Birinchi “Umumiy holat”.

Ushbu “Umumiy” holatda P_2 va P_3 quvlovchilarning vazifasi avvalambor qochuvchini bir joyda turib qolmasligi, doimo harakatlantirib turishdan va qochuvchi uchun harakatlangani sikl qoldirmaslikdan iborat. t_1 vaqtdan boshlab P_1 quvlovchi E qochuvchini l_1 to'g'ri chiziqqa keltirmaydi, ya'ni P_1 quvlovchi qochuvchi bilan l_1 to'g'ri chiziqqa nisbatan simmetrik harakat qiladi. Shuning uchun E qochuvchi uchun $D_3C_4C_3B_3B_4A_3B_5B_6C_6C_5$ sikl mavjud bo'ladi (5-rasm). Qochuvchi aynan shu sikldan chiqq olmay, faqat shu joyda harakatlanadi. Aynan

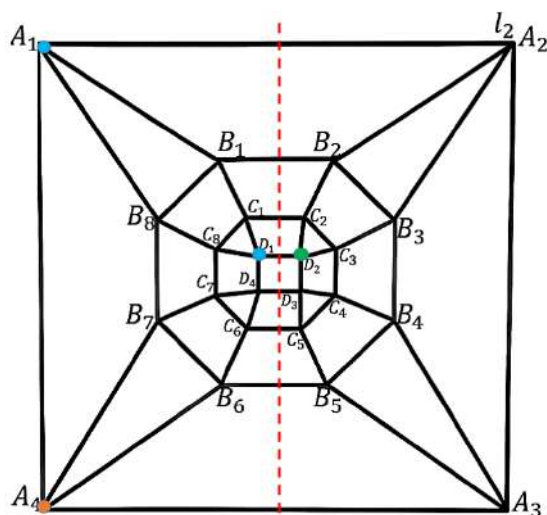
shu paytda P_2 va P_3 quvlovchilar bir-biriga simmetrik ravishda harakatlanib mos ravishda A_2 va A_3 uchlarni egallab olishadi. Natijada, qochuvchining $B_4A_3B_5$ sikli daraxtga aylanib qoladi va endi u $D_3C_4C_3B_3B_4B_5B_6C_6C_5$ siklda harakatlanishga majbur bo'ladi. P_2 va P_3 quvlovchilar esa mos ravishda $A_2B_3B_4$ va A_3B_5 qirralar bo'ylab harakatini davom ettirib, qochuvchining $C_4C_3B_3B_4B_5B_6C_6C_5$ siklini, B_4C_4 va B_5C_5 qirralari bo'ylab harakatlanib esa $D_3C_4C_5$ siklni daraxtga aylantirib uni qamalga olishni boshlaydilar (5-rasm).



5-rasm. Qochuvchining qamalga olinishi.

Shunday qilib, qochuvchi P_1, P_2, P_3 quvlovchilarning biriga albatta tutiladi. Maksimal holatda qochuvchi $t = 7 + \frac{6}{p_3}$ vaqtdan so'ng P_2 yoki P_3 quvlovchilardan biriga tutiladi. Bu bilan birinchi "Umumiy holat" masalasi yakuniga yetdi.

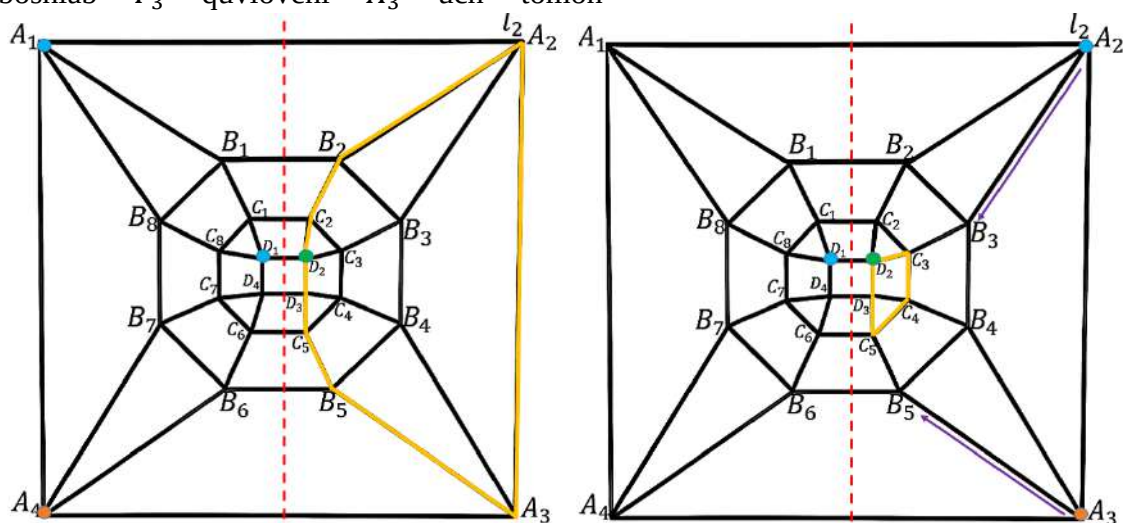
2. Ikkinchi "Umumiy holat"da quvlovchilar mos ravishda D_1, A_1, A_4 uchlarda qochuvchi esa D_2 uchda joylashgan bo'lsin (6-rasm). Bu holatda birinchi quvlovchining vaziyati $P_1(t_1)$ qochuvchining vaziyati $E(t_1)$ bilan l_2 to'g'ri chiziqqa nisbatan simmetrik joylashgan bo'ladi.



6-rasm. Ikkinchi "Umumiy holat".

Ikkinchi "Umumiy holat"da ham birinchi quvlovchining vazifasi qochuvchini l_2 to'g'ri chiziqqa keltirmaydi. Shuning uchun qochuvchi uchun harakatlangani faqat bitta $D_2C_2B_2A_2A_3B_5C_5D_3$ sikl qoladi. t_1 vaqtdan boshlab P_3 quvlovchi A_3 uch tomon

harakatlanadi. $t_2 = t_1 + \frac{1}{\rho_3}$ vaqtdan boshlab P_2 quvlovchi A_2 uch tomonga qarab harakatlanadi. Natijada qochuvchi $D_2C_3C_4C_5D_3$ sikldagina harakatlana oladi. (7-rasm).

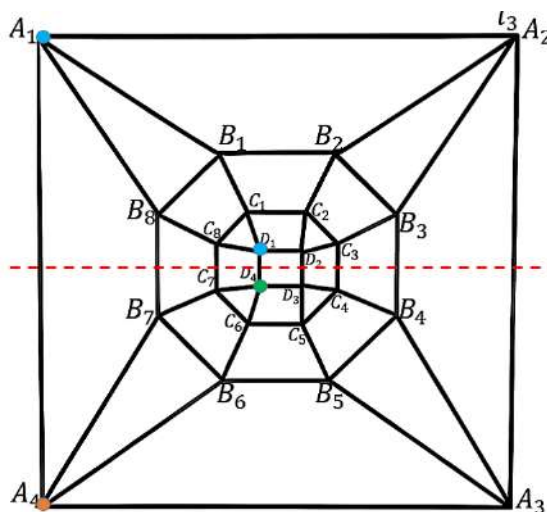


7-rasm. Qochuvchining qamalga olinishi

P_2 quvlovchi qanchadir vaqtdan keyin B_5 qilishda davom etishadi. Qochuvchi C_2 uchga yetib kelgandan so'ng P_3 quvlovchi B_3, B_2, C_5 va D_3 uchlarga kela olmaydi, chunki uch tomon harakatlanib qochuvchini taqib P_1 quvlovchi qochuvchi bilan simmetrik

harakat qilib qochuvchini shu nuqtalarda tuta oladi. Natijada, taqdirga tan bergan qochuvchi maksimal $t = t_1 + \frac{1}{\rho_3} + 3 + \frac{1}{\rho_3} = \frac{6}{\rho_3} + 6$ vaqtda quvuvchilarning biri tomonidan tutiladi. Bu bilan ikkinchi “Umumiy holat” ham yakunlandi.

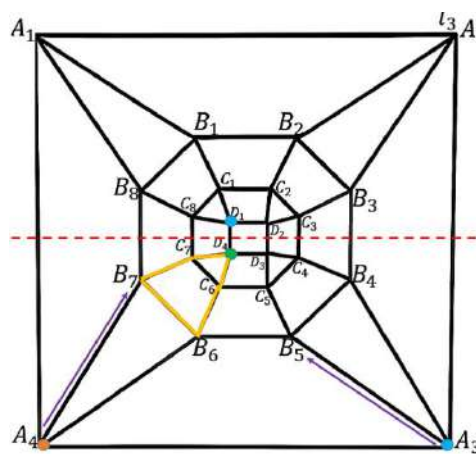
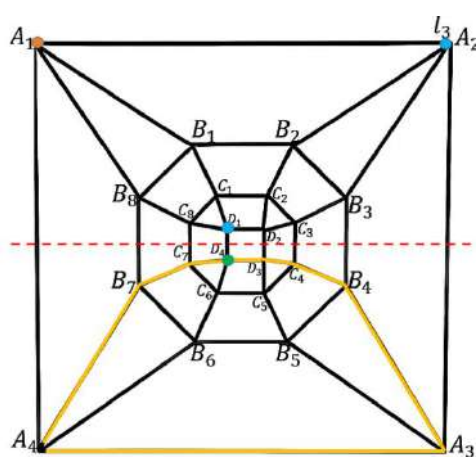
3. So’nggi uchinchi “Umumiy holat” ikkinchi “Umumiy holat”ga o’xshashib ketadi. Endi quvuvchilar D_1, A_1, A_2 , bu safar qochuvchi D_4 uchda bo’lsin (8-rasm). Bu holatda birinchi quvlovchining vaziyati $P_1(t_1)$ qochuvchining vaziyati $E(t_1)$ bilan l_3 to’g’ri chiziqqa nisbatan simmetrik joylashgan bo’ladi.



8-rasm. Uchinchi “Umumiy holat”.

Uchinchi “Umumiy holat”da quvlash jarayoni huddi ikkinchi “umumiy holat”dagidek kechadi. P_1 quvlovchi qochuvchini l_3 to’g’ri chiziqqa keltirmaydi. Qochuvchi uchun bitta $D_4 D_3 C_4 B_4 A_3 A_4 B_7 C_7$ sikl mavjud. P_2 va P_3 quvlovchilar $D_4 D_3 C_4 B_4 A_3 A_4 B_7 C_7$ siklni daraxtga aylantirishga harakat qiladi. Buning uchun t_1 vaqtdan boshlab P_2 quvlovchi A_3 uch tomon P_3 quvlovchi A_4 uch tomon harakatlanadi va qochuvchi $D_4 C_6 B_6 B_7 C_7$ sikldagina yura olishi mumkin. $t_2 = t_1 + 1 + \frac{1}{\rho_3}$ vaqtdan boshlab P_2 va P_3 quvlovchilar mos

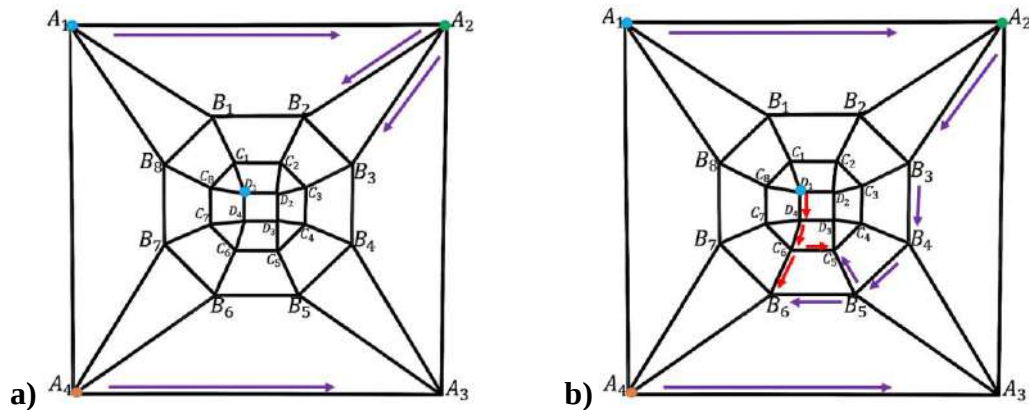
ravishda B_5 va B_7 uchlari tomonga qarab harakatlanadi. Quvlovchilar B_5 va B_7 uchlarga yetib kelgandan so’ng qochuvchi qamalga tushib qoladi. (9-rasm). Birinchi quvlovchi esa har doimgidek l_3 to’g’ri chiziqqa nisbatan qochuvchi bilan simmetrik harakatlanadi. Natijada, qochuvchi maksimal $t = t_1 + 1 + \frac{1}{\rho_3} + 1 + \frac{1}{\rho_3} = \frac{6}{\rho_3} + 5$ vaqtda quvuvchilarning biri tomonidan tutiladi. Bu bilan uchinchi “Umumiy holat” ham yakunlandi.



9-rasm. Qochuvchining qamalga olinishi.

Endi qochuvchi D_2, D_3 va D_4 uchlardan boshqa uchlarda bo'lgan xususiy hollarni o'rganamiz. Bu xususiy hollarning hammasini ham o'rganish shart emas. Chunki quvuvchilar shunday joylashib olganki, ba'zi hollar o'zaro simmetrik ko'rinishda bo'ladi. Qochuvchining $B_1B_2B_3B_4B_5B_6B_7B_8$ va $C_1C_2C_3C_4C_5C_6C_7C_8$ sakkizburchakdan tashqari bo'lgan hollarini ko'raylik. Masalan, qochuvchi A_2 uchda

bo'lgan holni ko'raylik chunki qochuvchi P_2 va P_3 quvlovchilarning taqibidan qochish davomida va "Umumiy holat"larga tushmaslik uchun A_2 va A_3 uchlarning biridan albatta o'tadi va qochuvchi A_2 uchda bo'lgan hol qochuvchi A_3 uchda bo'lgan hol bilan o'zaro simmetrik bo'lgani uchun biz qochuvchining A_2 uchda bo'lgan holatini o'rgansak kifoya (10a-rasm).



10-rasm.

P_2 va P_3 quvlovchilar qochuvchini harakatga keltirish uchun A_2 va A_3 uchlarga tomon harakatlarni boshlaydi. P_1 quvlovchi esa D_1 uchda qoladi. A_2 uchda turgan qochuvchi uchun harakatlangani 3 xil imkoniyat bor: 1) A_1 uch tomonga bora olmaydi, chunki bu tomondan P_2 quvlovchi kelyabdi; 2) B_2 uch tomonga harakatlanib bu uchga yetib kelgandan so'ng B_1, B_3 yoki C_2 uchlarga tomon yursa quvlovchilar guruhi qochuvchini tutib oladi; 3) B_3 uchga kelganda B_2 va C_3 uchlarga tomon harakatlangan tutiladi, B_4 uchga borsa P_2 quvlovchi B_3 uchga keladi va B_4 uchdan C_4 uchga kelsa tutiladi, B_4 dan B_5 ga kelsa P_2 quvlovchi B_4 uch tomon yuradi va qochuvchi B_6 va C_5 uchga kelganda quvlovchilar tomonidan tutiladi va o'yin quvlovchilar foydasiga hal bo'ladi (10b-rasm);

va bu sakkizburchaklarda qolishni istaydi. Masalan, qochuvchi sakkizburchaklardagi qaysidir qirrada tinch turadigan bo'lsa, P_2 va P_3 quvlovchilar eng qisqa masofalar orqali bu sakkizburchaklarga yetib kelishadi va qochuvchini sakkizburchaklarga kiritadi. Qochuvchi sakkizburchaklarga kirish jarayonida D_2, D_3 va D_4 uchlarning biridan o'tishga majbur. Bu esa qochuvchi 3 "Umumiy holat"larning biriga tushib qoladi va quvlovchilarning biri tomonidan tutiladi.

Qochuvchi $B_1B_2B_3B_4B_5B_6B_7B_8$ va $C_1C_2C_3C_4C_5C_6C_7C_8$ sakkizburchaklarda bo'lsa, u bu sakkizburchaklarda ko'p harakat qila olmaydi. Boshqacha qilib aytadigan bo'lsak, u bu sakkizburchaklarda sikl qila olmaydi. Chunki birinchi quvlovchi bunga yo'l qo'ymaydi. Qochuvchi yuqoridagi hollarga ko'ra agar $B_1B_2B_3B_4B_5B_6B_7B_8$ va $C_1C_2C_3C_4C_5C_6C_7C_8$ sakkizburchaklardan tashqariga chiqadigan bo'lsa, tutilishini biladi

Shunday qilib, ikki P_2 va P_3 quvlovchilar qochuvchini domiy harakatda bo'lishga majbur qilishadi. P_1 quvlovchi ham doim D_1 uchda turavermaydi. Qochuvchining harakatiga qarab uning yo'nalishini cheklash uchun payt poylab turadi, ya'ni birinchi quvlovchining asosiy vazifasi l_1, l_2 va l_3 to'g'ri chiziqqa nisbatan qochuvchini simmetrik holatga keltirib olish. Natijada, qochuvchi qachondir albatta uchta "Umumiy holat"larga tushib qoladi va o'yin quvlovchilar guruhi foydasiga hal bo'ladi. Teorema isbotlandi.

Xulosa

Ushbu maqolada Arximed jismlar turiga mansub Rombikubo-oktaedrning

qirralarida uchta quvlovchi va bir qochuvchili differensial o'yin o'rganildi. Bunda ikki quvlovchi va qochuvchining maksimal tezliklari teng, umumiylikka ziyon yetkazmagan holda bu tezlikni 1 deb oldik. Uchinchi quvlovchining tezligini esa nomigagina mavjud bo'lsa kifoya deb oldik, ya'ni O'dan biroz katta bo'lsa kifoya.

Rombikubo-oktaedrning qirralarida bir qochuvchini tutish uchun eng kamida 3 quvlovchi yetarli ekan. O'yin davomida quvlovchilarga o'yinni yakunlash uchun strategiyalar qurildi. [7] maqola ushbu ishni o'rganishimiz uchun asos bo'ldi.

ADABIYOTLAR

- [1] Azamov A.A. (1983). On Pontryagin's second method in linear differential games of pursuit. Mat. Sb. (N.S.), 118(160):3(7) (1982), 422–430; Math. USSR-Sb., 46:3, 429–437.
- [2] Satimov N.Y., Rikhsiev B.B. (2000). Methods of Solving of Evasion Problems in Mathematical Control Theory, Fan, Tashkent, Uzbekistan.
- [3] Ibragimov G.I., Ferrara M., Ruziboev M., Pansera B.A. (2021). Linear evasion differential game of one evader and several pursuers with integral constraints. International Journal of Game Theory. doi.org/10.1007/s00182-021-00760-6
- [4] Ibragimov G.I., Salleh Y. (2012). Simple motion evasion differential game of many pursuers and one evader with integral constraints on control functions of players. Journal of Applied Mathematics, 2012, Article ID 748096, doi:10.1155/2012/748096.
- [5] Ibragimov G.I. (2023). Evasion differential game of one evader and many slow pursuers. Dynamic Games and Applications. https://doi.org/10.1007/s13235-023-00501-2.
- [6] Azamov A.A., Ibragimov G.I., Ibaydullaev T., Alias I.A. (2021) Optimal number of pursuers in the differential games on the 1-skeleton of orthoplex. Symmetry (Game Theoretical Symmetry Dynamic Processes). 10.3390/sym13112170.
- [7] Ibragimov G.I., Xasanjonova N.U. (2023) Kesik tetraedrda ikki quvlovchi va bir qochuvchining differensial o'yini. Scientific Bulletin of NamSU. № 7: 28–36.
- [8] Qo'shaqov X.Sh, Muhammadjonova M.A, Maxammadzokirov Sh.Sh. (2024) Differential game of three pursuers and one evader on an Elongated square Gyrobicupola. Scientific Bulletin of ASU. № 1(6): 49-53.

О единственности решения трехмерной задачи Франкля для уравнения смешанного типа с сингулярным коэффициентом

К.Т. Каримов^{1,2}, А.М. Шокиров¹

¹Ферганский государственный университет,

²Институт математики имени В.И. Романовского Академии наук Республики Узбекистан

karimovk80@mail.ru, asrorshokirov87@gmail.com

Аннотация. В работе исследована задача Франкля для трехмерного уравнения смешанного типа с сингулярным коэффициентом. Методом спектрального анализа доказана единственность решения поставленной задачи в классе регулярных решений.

Ключевые слова: Задача Франкля, уравнение смешанного типа, сингулярный коэффициент, единственность решения, спектральный метод, трёхмерная область, собственные значения, собственные функций.

Annotatsiya. Maqolada uch o'lchovli singulyar koeffitsientli aralash tipdagi tenglama uchun Frankl masalasi tadqiq qilingan. Spektral analiz usuli yordamida masalaning yagona yechimi borligi regular yechimlar sinfida isbotlangan.

Kalit so'zlar: Frankl masalasi, aralash tipdagi tenglama, singular koeffitsient, yechimning yagonaligi, spektral usul, uch o'lchovli soha, xos qiymatlar, xos funksiyalar.

Abstract. In this paper, the Frankl problem for a three-dimensional mixed-type equation with a singular coefficient is investigated. The uniqueness of the solution to the problem in the class of regular solutions is proved by the spectral analysis method.

Keywords: Frankl problem, mixed-type equation, singular coefficient, uniqueness of solution, spectral method, three-dimensional domain, eigenvalues, eigenfunctions.

1. Введение. Первые результаты по задаче Франкля для уравнений Лаврентева-Бицадзе $u_{xx} + (\operatorname{sgn} y)u_{yy} = 0$ и Чаплыгина

$K(y)u_{xx} + u_{yy} = 0, K(0) = 0, K'(y) > 0$ получены А.В.Бицадзе [1], [2].

В работах Ю.В.Девингтала [3], [4], [5] изучена задача Франкля для уравнения Чаплыгина при

$K(y) \in C^1(\bar{D}), K(-y) = -K(y)$, а Е.И. Моисеев в работе [6] методом спектрального анализа построил решение задачи Франкля, для уравнения

$$u_{xx} + \operatorname{sgn} y \cdot u_{yy} - \mu^2 \operatorname{sgn}(x + y)u = 0, \mu = \operatorname{const}.$$

Также своими учениками [7], [8], [9], [10], [11] изучил вопросы полноты, базисности системы собственных функций в области эллиптичности видоизмененной задачи

Франкля с нелокальными условиями первого и второго родов.

После этих работ возрос интерес к изучению задачи Франкля. Отметим, что типа таких задач была объектом исследования работ [13], [14], [14], [15]. В этих работах рассмотрены некоторые обобщения задачи Франкля или аналоги задачи Франкля или в специальных областях.

Перечисленные работы, посвященные данной проблеме, преимущественно затрагивают двумерные задачи. Трёхмерные задачи рассматриваются в работах [16], [17], при этом задача Франкля для трёхмерных уравнений смешанного типа с сингулярными коэффициентами до сих пор остается малоизученной.

2. Постановка задачи. В настоящем работе уравнения

$$\operatorname{sign}(x+y)\left[U_{xx}+\operatorname{sign} y \cdot U_{yy}\right]+U_{zz}+\frac{2 \gamma}{z} U_z=0 \quad (1)$$

рассматриваются в области $D=\Omega \times(0, c)$, где Ω –конечная односвязная область плоскости xOy , ограниченная дугой $\bar{\sigma}_0$ и отрезками $\overline{MM^*}$, $\overline{M^*P}$, где $\bar{\sigma}_0=\left\{(x, y): x^2+y^2=1, x \geq 0, y \geq 0\right\}$, $MM^*=\left\{(x, y): x=0,-1< y < 1\right\}$, $M^*P=\left\{(x, y): x-y=1, 0< x < 1\right\}$.

Введем обозначения: $\Omega_0=\Omega \cap(y>0)$, $\Omega_1=\Omega \cap\left\{(x, y): y<0, x+y>0\right\}$, $\Omega_2=\Omega \cap\left\{y<0, x+y<0\right\}$, $D_j=\Omega_j \times(0, c)$, $j=\overline{0,2}$, $OP=\Omega \cap(y=0)$, $OQ=\left\{(x, y): x+y=0, 0< x < 1 / 2\right\}$, $OM=\left\{(x, y): x=0, 0< y < 1\right\}$, $OM^*=\overline{MM^*} \setminus \overline{OM}$, $O(0,0)$, $M(0,1)$, $M^*(0,-1)$, причем здесь $\gamma=\operatorname{const} \in R$, $0<\gamma<1 / 2$.

В области D уравнения (1) принадлежит смешанному типу, а именно: в области D_0 - эллиптическому типу, а в областях D_1 и D_2 - гиперболическому типу. Прямоугольник $OP \times(0, c)$ являются плоскостями изменения типа, а $z=0$ плоскостями сингулярности коэффициента уравнения.

Для уравнения (1) в области Ω исследуем следующую задачу:

Задача $\mathcal{F}^{(1)}$. Найти функцию $U(x, y, z)$, удовлетворяющую в области D уравнению (1) и следующим условиям:

$$U(x, y, z) \in C(\bar{D}) \cap C_{x, y, z}^{2,2,2}\left(D_0 \cup D_1 \cup D_2\right), U_x \in C(\bar{D}), \quad (2)$$

$$U(x, y, z)=0, (x, y) \in \bar{\sigma}_0, z \in(0, c), \quad (3)$$

$$U(0, y, z)=0, y \in[-1,1], z \in(0, c), \quad (4)$$

$$U_x(0, y, z)+q_1 U_x(0,-y, z)=0, y \in(-1,0), z \in[0, c], \quad (5)$$

$$U(x, y, 0)=f_1(x, y), U(x, y, c)=f_2(x, y), (x, y) \in \bar{\Omega}, \quad (6)$$

а также условию склеивания

$$\lim _{y \rightarrow-0} U_y(x, y, z)=\lim _{y \rightarrow+0} U_y(x, y, z), x \in(0,1), z \in(0, c), \quad (7)$$

где q_1 – заданное число, а $f_1(x, y)$ и $f_2(x, y)$ – заданные функции.

Очевидно, что задача $\mathcal{F}^{(1)}$ является задачей типа задачи Франкля [18], [19]. Если $q_1=0$, то из (5) следует, что $U_x(0, y, z)=0, y \in(-1,0), z \in[0, c]$. Тогда, принимая во внимание это и (4), на основании единственности решения задачи Коши [20] для уравнения (1) в области D_2 , имеем $U(x, y, z) \equiv 0, (x, y, z) \in \bar{D}_2$, откуда следует, что

$U(x, y, z)|_{\overline{OQ} \times [0, c]} = 0$. Следовательно, в этом случае задача $\mathcal{F}^{(1)}$ эквивалентна задаче Трикоми для уравнения (1) в области $D_0 \cup [OP \times (0, c)] \cup D_1$. Задача Трикоми для уравнения с сингулярными коэффициентами исследованы в работах [21], [22].

2. Единственность решение задачи $\mathcal{F}^{(1)}$

Находим нетривиальные решения уравнения (1), удовлетворяющие условиям (2)-(5) и (7). Разделив переменные по формуле $U(x, y, z) = u(x, y)Z(z)$, из уравнения (1) и условий (2)-(5), (7) получим уравнение

$$Z''(z) + \frac{2\gamma}{z} Z'(z) - \lambda Z(z) = 0, \quad 0 < z < c, \quad (8)$$

и следующая задача на собственные значения:

$$u_{xx} + (\operatorname{sgn} y)u_{yy} + \lambda \operatorname{sgn}(x+y)u = 0, \quad (x, y) \in \Omega_0 \cup \Omega_1 \cup \Omega_2, \quad (9)$$

$$u(x, y) = 0, \quad (x, y) \in \bar{\sigma}_0, \quad (10)$$

$$u(0, y) = 0, \quad y \in [-1, 1], \quad (11)$$

$$u_x(0, y) + q_1 u_x(0, -y) = 0, \quad y \in (-1, 0). \quad (12)$$

Сначала исследуем спектральную задачу (9)-(12). В работе [23] найдено, что собственные функции спектральной задачи (9)-(12) в виде

$$u_{n,m}^{(j)}(x, y) = \begin{cases} c_{n,m}^{(j)} \sin \left[\omega_{j,n} \left(\frac{\pi}{2} - \varphi \right) \right] J_{\omega_{j,n}} \left[\alpha_{n,m}^{(j)} \sqrt{x^2 + y^2} \right], & (x, y) \in \Omega_0; \\ \frac{1}{2} c_{n,m}^{(j)} \left[q_1 \left(\frac{x-y}{x+y} \right)^{\omega_{j,n}/2} - (-1)^j \sqrt{2-q_1^2} \left(\frac{x+y}{x-y} \right)^{\omega_{j,n}/2} \right] \times \\ \times J_{\omega_{j,n}} \left[\alpha_{n,m}^{(j)} \sqrt{x^2 - y^2} \right], & (x, y) \in \Omega_1; \\ \frac{1}{2} q_1 c_{n,m}^{(j)} \left[\left(\frac{y-x}{y+x} \right)^{\omega_{j,n}/2} - \left(\frac{y+x}{y-x} \right)^{\omega_{j,n}/2} \right] J_{\omega_{j,n}} \left[\alpha_{n,m}^{(j)} \sqrt{y^2 - x^2} \right], & (x, y) \in \Omega_2, \\ j = \overline{1, 2}; \quad n, m = 1, 2, \dots \end{cases} \quad (13)$$

Здесь $\omega_{1,n} = 4(n-1) + \frac{1}{2} + \frac{2}{\pi} \arccos \left(\frac{q_1}{\sqrt{2}} \right)$, $-\sqrt{2} \leq q_1 \leq \sqrt{2}$,

$$\omega_{2,n} = \begin{cases} 4(n-1) + \frac{1}{2} - \frac{2}{\pi} \arccos\left(\frac{q_1}{\sqrt{2}}\right), & \text{если } 1 \leq q_1 \leq \sqrt{2}, \\ 4n + \frac{1}{2} - \frac{2}{\pi} \arccos\left(\frac{q_1}{\sqrt{2}}\right), & \text{если } -\sqrt{2} \leq q_1 \leq 1, \end{cases}$$

$J_l(x)$ – функция Бесселя порядка l первого и второго рода [24], $\alpha_{n,m}^{(1)}$ и $\alpha_{n,m}^{(2)}$ – соответственно m -ые положительные корни уравнений $J_{4n-3}(\sqrt{\lambda}) = 0$ и $J_{4n}(\sqrt{\lambda}) = 0$, $c_{n,m}^j$ – произвольные постоянные, $\lambda_{n,m}^j = \alpha_{n,m}^j$, $\lambda_{nm} = \lambda_{n,m}^{(1)} + \lambda_{n,m}^{(2)}$.

Для системы (13) справедлива следующая теорема.

Теорема [23]. Система собственных функций

$$\left\{u_{n,m}(x,y)\right\}_{n,m=1}^{\infty} = \left\{u_{n,m}^{(1)}(x,y)\right\}_{n,m=1}^{\infty} \cup \left\{u_{n,m}^{(2)}(x,y)\right\}_{n,m=1}^{\infty} \quad \text{задачи (9)-(12) при } q_1 = -1,$$

где $\left\{u_{n,m}^{(j)}(x,y)\right\}_{n,m=1}^{\infty}$ – функции, определяемые равенством (13), полна в пространстве $L_2(\Omega_0)$.

Полагая в уравнении (8) $\lambda = \lambda_{nm}$, найдем его общее решение:

$$Z_{nm}(z) = c_1 z^{1/2-\gamma} I_{1/2-\gamma}(\sqrt{\lambda_{nm}} z) + c_2 z^{1/2-\gamma} K_{1/2-\gamma}(\sqrt{\lambda_{nm}} z), \quad z \in [0, c], \quad (14)$$

где c_1 и c_2 – произвольные постоянные.

Теперь докажем единственность решения задачи $\mathfrak{F}^{(1)}$. Пусть $U(x,y,z)$ – решение задачи $\mathfrak{F}^{(1)}$. Рассмотрим следующую функцию:

$$\omega_{nm}(z) = \iint_{\Omega_0} U(x,y,z) u_{n,m}(x,y) dx dy, \quad n, m \in N, \quad (15)$$

$$\text{где } \left\{u_{n,m}(x,y)\right\}_{n,m=1}^{\infty} = \left\{u_{n,m}^{(1)}(x,y)\right\}_{n,m=1}^{\infty} \cup \left\{u_{n,m}^{(2)}(x,y)\right\}_{n,m=1}^{\infty}.$$

На основании (15) введем функции

$$\omega_{nm\varepsilon}(z) = \iint_{\Omega_0\varepsilon} U(x,y,z) u_{n,m}(x,y) dx dy, \quad n, m \in N, \quad (16)$$

где ε – достаточно малые положительные числа.

$$\text{Очевидно, что } \lim_{\varepsilon \rightarrow 0} \omega_{nm\varepsilon}(z) = \omega_{nm}(z).$$

С помощью функции (16) и уравнения (1), вычислим выражение $B_{\gamma-1/2}^z \omega_{nm\varepsilon}(z)$:

$$\begin{aligned} B_{\gamma-1/2}^z \omega_{nm\varepsilon}(y) &= \iint_{\Omega_{0\varepsilon}} \left(B_{\gamma-1/2}^z U(x, y, z) \right) u_{n,m}(x, y) dx dy = \\ &= - \iint_{\Omega_{0\varepsilon}} \operatorname{sgn}(x+y) \left[U_{xx} + \operatorname{sgn} y \cdot U_{yy} \right] u_{n,m}(x, y) dx dy. \end{aligned} \quad (17)$$

Применяя правило интегрирования по частям, и переходя к пределу при $\varepsilon \rightarrow 0$ и учитывая условия задачи $\mathfrak{F}^{(1)}$, получим равенство

$$\left[\omega_{nm}(z) \right]'' + \frac{2\gamma}{z} \left[\omega_{nm}(z) \right]' - \lambda_{nm} \omega_{nm}(z) = 0, \quad z \in (0, c).$$

Следовательно, функция $\omega_{nm}(z)$ удовлетворяет дифференциальному уравнению (8) при $\lambda = \lambda_{nm}$.

Кроме того, в силу граничных условий (6), из (15) следует, что функция $\omega_{nm}(z)$ удовлетворяет следующим граничным условиям:

$$\omega_{nm}(0) = f_{1nm}, \quad \omega_{nm}(c) = f_{2nm}, \quad (18)$$

где

$$f_{lnm} = \iint_{\Omega_0} f_l(x, y) u_{n,m}(x, y) dx dy, \quad n, m \in N, \quad l = \overline{1, 2}. \quad (19)$$

Отсюда следует, что функция $\omega_{nm}(z)$, которая имеет вид (15), является решением уравнения (8) при $\lambda = \lambda_{nm}$, удовлетворяющим условиям (18). Поэтому, удовлетворяя общее решение (14) уравнения (8) (при $\lambda = \lambda_{nm}$) условиям (18), однозначно находим $\omega_{nm}(y)$:

$$\omega_{nm}(z) = P_{nm}(z) f_{2nm} + \left[\bar{K}_{1/2-\gamma}(\sqrt{\lambda_{nm}} z) - P_{nm}(z) \bar{K}_{1/2-\gamma}(\sqrt{\lambda_{nm}} c) \right] f_{1nm}, \quad (20)$$

где $P_{nm}(z) = \left(\frac{z}{c} \right)^{1/2-\gamma} \frac{I_{1/2-\gamma}(\sqrt{\lambda_{nm}} z)}{I_{1/2-\gamma}(\sqrt{\lambda_{nm}} c)}, \quad \bar{K}_\nu(z) = \frac{2^{1-\nu} z^\nu K_\nu(z)}{\Gamma(\nu)}, \quad \nu > 0.$

Из изложенного выше следует, что каждая из функций

$$U_{nm}(x, y, z) = \omega_{nm}(z) u_{n,m}(x, y), \quad n, m \in N, \quad (x, y, z) \in \Omega_0, \quad (21)$$

где $\omega_{nm}(z)$ и $u_{n,m}(x, y)$ -функции, определяемые соответственно равенствами (20) и (13) удовлетворяют условиям поставленной задачи.

Теперь можем доказать следующую теорему.

Теорема. Если существует решение задачи $\mathfrak{F}^{(1)}$, то оно единственно.

Доказательство. Для этого достаточно доказать, что однородная задача $\mathfrak{F}^{(1)}$ имеет только тривиальные решения. Пусть $f_1(x, y) = f_2(x, y) \equiv 0$, тогда согласно равенствам (19), имеем $f_{1nm} = f_{2nm} = 0$ при всех $n, m \in N$. На основании этого, из (20) и (15) вытекает, что

$\iint_{\Omega_0} U(x, y, z) u_{n,m}(x, y) dx dy = 0, n, m \in N$. Отсюда, в силу полноты системы собственных

функций $\{u_{n,m}(x, y)\}_{n,m=1}^{\infty} = \{u_{n,m}^{(1)}(x, y)\}_{n,m=1}^{\infty} \cup \{u_{n,m}^{(2)}(x, y)\}_{n,m=1}^{\infty}$ в пространстве $L_2(\Omega_0)$ и $u(x, y, z) \in C(\bar{\Omega}_0)$, следует, что $U(x, y, z) \equiv 0$ в $\bar{D}_0$.

Пользуясь этим равенством, легко убедиться, что $U(x, +0, z) \equiv 0, U_y(x, 0, z) \equiv 0, x \in [0, 1], z \in [0, c]$, а также условий (4), (5), имеем $U(x, -0, z) \equiv 0, U_y(x, -0, z) \equiv 0, U(-0, y, z) \equiv 0, U_x(-0, y, z) \equiv 0$. В силу единственности решения задачи Коши в областях D_1 и D_2 , следует $U(x, y, z) \equiv 0, (x, y, z) \in \bar{D}_1 \cup \bar{D}_2$. Теорема доказана.

Литература

1. Бицадзе А.В. Об одной задаче Франкля. Доклады АН СССР, 109:6 (1956), 1091-1094.
2. Бицадзе А.В. О единственности решения задачи Франкля для уравнения Чаплыгина. Доклады АН СССР, 112:3 (1957), 375-376.
3. Девингталь Ю.В. О существовании решения одной задачи Ф.И. Франкля. Доклады АН СССР, 119:1. 1958. -С.15-18.
4. Девингталь Ю.В. О существовании и единственности решения одной задачи Ф.И.Франкля. Известия вузов. Математика, №2 (3), -С. 39-51.
5. Девингталь Ю.В. К вопросу о существовании и единственности решения задачи Франкля. УМН, 14:1(85). 1959. -С. 177-182.
6. Моисеев Е.И. О решении задачи Франкля в специальной области. Дифференциальные уравнения, 28:4. 1992. -С. 721-723.
7. Moiseev, E.I., Abbasi, N. On the completeness of eigenfunctions of the Frankl problem with the oddness condition. Diff Equat 44, 408–412 (2008). <https://doi.org/10.1134/S0012266108030117>
8. Moiseev, E.I., Abbasi, N. On the basis property of eigenfunctions of the Frankl problem with a nonlocal parity condition. Diff Equat 44, 855–860 (2008). <https://doi.org/10.1134/S0012266108060098>
9. Moiseev, E.I., Abbasi, N. Basis property of the eigenfunctions of a generalized gasdynamic Frankl problem with a nonlocal evenness condition and with a discontinuity in the solution gradient. Diff Equat 45, 1485–1490 (2009). <https://doi.org/10.1134/S0012266109100115>
10. Moiseev, E.I., Ambartsumyan, V.E. On the basis property of eigenfunctions of the Frankl problem with

- a nonlocal parity condition of the second kind. Diff Equat 45, 1769–1774 (2009). <https://doi.org/10.1134/S0012266109120076>
11. Moiseev, E.I., Ambartsumyan, V.E. On the basis property of the eigenfunctions of the Frankl problem with nonlocal evenness and oddness conditions of the second kind. Dokl. Math. 81, 429–433 (2010). <https://doi.org/10.1134/S1064562410030257>
12. Бурмистров Б.Н. О некоторых краевых задачах типа задачи Франкля для систем уравнений 1-го порядка смешанного типа. Дис. ... канд.физ.-матем. наук, Казань, 1970.
13. Zhegalov V.I. The Frankl problem with a shift. Soviet Math., 23:9. 1979. pp. 10-19.
14. Клоков В.В. О единственности решения обобщенной задачи Франкля. Тр. сем. по краев. задачам, 3. Изд-во Казанского университета, Казань. 1966. -С.66-67.
15. Майоров И.В. Об одной задаче Франкля для нелинейного уравнения смешанного типа. Дифференциальные уравнения, 4:1. 1968. -С. 46-51.
16. Салахитдинов М. С., Исломов Б. О трехмерном аналоге задачи Трикоми для уравнения смешанного типа", Докл. АН СССР, 311:4 (1990), 797–801
17. Moiseev E.I., Nefedov P.V. Frankl problem for the Lavrent'ev-Bitsadze equation in a 3D-domain // Integral Transforms and Special Functions, 2013, Vol. 24, Issue 7. P. 554-560.
18. Смирнов М.М. Уравнения смешанного типа. – М.: Наука, 1970. -296 с.
19. Франкль Ф.И. Обтекание профилей газом с местной сверхзвуковой зоной, оканчивающейся прямым скачком уплотнения//Прикладная

- математика и механика, Т.20, №2, 1956. –С. 196-202.
20. Тихонов А.Н., Самарский А.А. Уравнения математической физики. М.: Изд-во МГУ, 1999.
21. Karimov K.T., Shokirov A.M. A problem for a three-dimensional equation of mixed type with singular coefficient//International Journal of Applied Mathematics. Volume 37 No. 1 2024, pp. 73–92. <http://dx.doi.org/10.12732/ijam.v37i1.7>
22. Urinov A.K., Karimov K.T., Trikom problem for a three-dimensional equation of mixed type with three singular coefficients, *Bulletin of NUUz (Tashkent)*, **2** (2016), 14-25.
23. Салахитдинов М.С., Уринов А.К. К спектральной теории уравнений смешанного типа. –Ташкент: Mumtoz So'z. –2010. -355 с.
24. Ватсон Г. Н. Теория бесселевых функций. - М.:Т.1.Изд. ИЛ. –1949. -798 с.

YUKLANGAN ISSIQLIK TARQALISH TENGLAMASI UCHUN BIR CHEGARAVIY MASALA

Muzaffar Azizov¹, Qizlarxon Orifjonova¹

¹Farg'ona Davlat Universiteti, Murabbiylar ko'chasi.19, 150100, Farg'ona O'zbekiston

Annotatsiya. Ushbu maqolada yuklangan issiqlik tarqalish tenglamasi uchun bir chegaraviy masala o'rganiladi. Tadqiqotda boshlang'ich va chegaraviy shartlar bilan berilgan masalalar tahlil qilinib, masala yechimining mavjudligi va yagonaligi isbotlanadi.

Kalit so'zlar: Furiye usuli, Bessel tengsizligi, oddiy differensial tenglama, mojarant qator, variatsiyalash usuli, to'la ortonormal sistema.

Аннотация. В данной статье рассматривается одна краевая задача для нагруженного уравнения теплопроводности. В исследовании анализируются задачи, заданные начальными и граничными условиями, и доказываются существование и единственность решения задачи.

Ключевые слова: Ряд Фурье, неравенство Бесселя, обыкновенное дифференциальное уравнение, мажорантный ряд, вариационный метод, полная ортонормированная система.

Abstract. This paper studies a boundary value problem for the loaded heat conduction equation. Initial and boundary conditions are analyzed, and the existence and uniqueness of the solution are proved.

Keywords: Fourier series, Bessel inequality, ordinary differential equation, majorant series, variational method, complete orthonormal system.

I.Kirish

Parabolik tipdagi xususiy hosilali differensial tenglamalar ichida eng ko'p o'rganilgan va amaliyotda keng qo'llaniladigan tenglama bu issiqlik tarqalish tenglamasidir. Bu tenglama ko'plab fizik jarayonlarni, jumladan, issiqlik taqalishini, diffuziya jarayonlarini, molekulyar harakatlarni matematik modellashtirishda asosiy vosita bo'lib xizmat qiladi. Issiqlik tenglamasi ko'rinishi jihatidan quyidagicha ifodalanadi:

$$u_t - a^2 u_{xx} = 0,$$

bu yerda $u(x, t)$ - fazoning berilgan nuqtasida va vaqt momentida haroratni ifodalovchi noma'lum funksiya, a^2 esa issiqlik o'tkazuvchanlik koeffitsientidir.

Mazkur tenglama dastlab J. Furiye tomonidan XIX asrda o'rganilgan bo'lib, u bu tenglama uchun qo'yilgan masalani yechishning asosiy yondashuvini qatorlar yordamida hal qilish usulini taklif qilgan. Keyinchalik bu masala Sturm-Luivill nazariyasi, Grin funksiyasi va boshqa analitik metodlar yordamida yanada rivojlantirildi.

Issiqlik tenglamasining klassik ko'rinishlari, odatda, quyidagicha boshlang'ich va Dirixle tipidagi chegaraviy shartlar bilan qo'llaniladi:

$$\begin{cases} u(x, 0) = \varphi(x), & 0 \leq x \leq l, \\ u(0, t) = 0, \quad u(l, t) = 0, & 0 \leq t \leq T. \end{cases}$$

bunday shartlar ostida tenglama korrekt qo'yilgan bo'lib, ya'ni mavjudlik, yagonalik va yechimning berilganlarga uzluksiz bog'liqligi kabi xossalarga ega bo'ladi.

Ushbu ishda klassik issiqlik tarqalish tenglamasining noan'anaviy o'zgartirilgan shakli ko'rib chiqiladi:

$$Lu \equiv u_t - a^2 u_{xx} = u_t(x, t_0), \quad (1)$$

bu yerda $t_0 \in (0, T]$ berilgan vaqt momenti. Tenglamaning o'ng tomonidagi .. ifoda bu masalani an'anaviy boshlang'ich-chegaraviy masaladan farq qiladi. Ya'ni, bu holatda tenglamaning o'zi vaqtga bog'liq ma'lum bir nuqtadagi hosila orqali aniqlanadi. Bu esa masalaning turg'unlik (barqarorlik) va korrektlik jihatidan tahlilini murakkablashtiradi.

Shu bois, bunday turdagi masalalar ilmiy adabiyotda "yuklangan tenglamalar", "yuklangan manbali tenglamalar" yoki "moslashtirilgan issiqlik tenglamalari" sifatida ham qaraladi. Ular, odatda, fizik yoki muhandislik modellarida berilgan vaqtda sodir bo'ladigan impuls yoki energiya ta'sirini modellashtirilayotgan hollarda uchraydi.

Xususan, oxirgi yillarda $u_t - a^2 u_{xx} = f(x, t)$ ko'rinishidagi tenglamalarning yuklangan xollariga qiziqish ortmoqda. Biz ko'rib chiqayotgan masalada o'ng tomon $u_t(x, t_0)$ tarzida aniqlangan bo'lib, bu uni klassik issiqlik tenglamasidan tubdan farqlantiradi.

Yuklangan tenglamalarning chuqur o'rganilgan ko'rinishlaridan biri Nakhushiev A.M., Kreferov A.G. tomonidan taqdim etilgan bo'lib [1], ular parabolik tipdagi tenglamalarda manbalarni yechimning vaqtga bog'liq hosilasi sifatida aniqlashgan. Bu yondashuv, ayniqsa, vaqt bo'yicha lokal ta'sirli yuklanishlarni modellashtirish uchun samarali hisoblanadi.

Keyinchalik, Borisov A.V., Kreferov A.G.lar [2] bu masalalarni Sobolev fazolarida tahlil qilib, ularning yechimlar mavjudligi va yagonaligi shartlarini isbotlashgan. Bu ishlar bizning masalamizdagi $u_t(x, t_0)$ kabi yuklangan atamalarni umumlashtirish uchun nazariy asos bo'lib xizmat qiladi.

Kasr tartibli yuklangan parabolik tipdagi tenglamalar bo'yicha Agarwal, R.P., Baltaeva, Z.A., Xasanov, A.K. [3] ishida tadqiq etganlar. Ular bu masalani integral tenglamaga keltirish orqali mavjudlik va yagonalikni isbotlashgan.

Ceretani, A., Tarzia, D.A., Villa, R. [4] tomonidan taklif etilgan ishlarda issiqlik oqimi chiziqli bo'lmagan funksiyaga bog'liq holda aniqlangan bo'lib, bu ularni noan'anaviy manbali tenglamalarga olib kelgan. Ularning ishlari bizning masalamizga metodologik yaqin, chunki ular ham vaqt bo'yicha bog'liq manbani integral ko'rinishda tahlil qilganlar.

Ushbu adabiy manbalar shuni ko'rsatadiki, parabolik tenglamalarning yuklangan xollari amaliy modellashtirishda, teskari masalalarda va signallarni qayta tiklashda muhim ahamiyat kasb etadi. Bizning masalamizda esa yuklangan ifoda vaqtning t_0 momentidagi hosilaga tenglashtirilgani uchun bu nafaqat klassik masaladan, balki aksariyat

yuklangan tenglamalardan ham farq qiladi. Shu sababli, ushbu ishda masalaga yangi yondashuvlar, ayniqsa, Furiye usuli va integral metodlar asosida yechimlar izlanadi.

II. Asosiy qism

Ushbu ishda yuklangan issiqlik tarqalish tenglamasi uchun chegaraviy masalani bayon qilamiz va o'rganamiz.

Ushbu $\Omega = \{(x, t) : 0 < x < l, 0 < t < T\}$ to'g'ri to'rtburchakda (1) tenglamani qaraylik, bu yerda l va T berilgan musbat haqiqiy sonlar, $u = u(x, t)$ - noma'lum funksiya, $t_0 \in (0, T]$.

Masalaning qo'yilishi.

Masala. $\bar{\Omega}$ sohada shunday $u(x, t) \in C_{x,t}^{1,0}(\bar{\Omega}) \cap C_{x,t}^{2,1}(\Omega)$ funksiya aniqlansinki, u (1) tenglamani va ushbu

$$u(x, 0) = \varphi(x), \quad 0 \leq x \leq l; \quad (2)$$

boshlang'ich shartni hamda

$$u(0, t) = 0, \quad u(l, t) = 0, \quad 0 \leq t \leq T \quad (3)$$

chegaraviy shartlarni qanoatlantirsin, bu yerda $\varphi(x)$ - berilgan ma'lum funksiya.

Masalalar yechimining yagonaligi.

1-teorema. Qo'yilgan masalaning yechimi mavjud bo'lsa, u yagonadir.

Isbot. Teskarisidan faraz qilamiz. Qo'yilgan masala $u_1(x, t)$ va $u_2(x, t)$ yechimlarga ega bo'lsin. U holda, ularning farqini

$$u_1(x, t) - u_2(x, t) = v(x, t). \quad (4)$$

deb olsak, $v(x, t)$ funksiya

$$Lv \equiv v_t - a^2 v_{xx} = v_t(x, t_0) \quad (5)$$

tenglamani hamda (2), (3) shartlarga mos bir jinsli shartlarni qanoatlantiradi.

Ma'lumki,

$$X_n(x) = \sqrt{\frac{2}{l}} \sin \lambda_n x, \quad n = 1, 2, \dots \quad (6)$$

sistema $L_2(0, l)$ fazoda to'la ortonormal sistemani tashkil qiladi.

Quyidagi funktsiyani qaraymiz:

$$d_n(t) = \int_0^l v(x, t) X_n(x) dx, \quad n = 1, 2, \dots \quad (7)$$

(7) funktsiyaga mos ravishda

$$d_{n,\varepsilon}(t) = \int_{\varepsilon}^{l-\varepsilon} v(x, t) X_n(x) dx, \quad 0 < \varepsilon < l,$$

$$(\varepsilon, l - \varepsilon) = \emptyset, \quad n = 1, 2, \dots \quad (8)$$

funktsiyani tuzib olamiz. Aniqki,

$$\lim_{\varepsilon \rightarrow 0} d_{n,\varepsilon}(t) = d_n(t), \quad t \in [0, T].$$

(8) tenglikni t bo'yicha bir marta differensiallab, quyidagi tenglikni hosil qilamiz:

$$d'_{n,\varepsilon}(t) = \int_{\varepsilon}^{l-\varepsilon} v_t(x, t) X_n(x) dx, \quad n = 1, 2, \dots$$

(5) tenglamadan foydalanib oxirgi tenglikni quyidagicha yozish mumkin:

$$d'_{n,\varepsilon}(t) = a^2 \int_{\varepsilon}^{l-\varepsilon} v_{xx}(x, t) X_n(x) dx + \int_{\varepsilon}^{l-\varepsilon} v_t(x, t_0) X_n(x) dx, \quad n = 1, 2, \dots \quad (9)$$

(9) tenglikning o'ng tarafidagi birinchi integralni ikki marta bo'laklab integrallab, so'ngra undan $\varepsilon \rightarrow 0$ da limitga o'tsak, quyidagi ifodaga ega bo'lamiz:

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} d'_{n,\varepsilon}(t) &= \\ &= a^2 \lim_{\varepsilon \rightarrow 0} [X_n(l - \varepsilon) v_x(l - \varepsilon, t) - X_n(\varepsilon) v_x(\varepsilon, t)] - \\ &- \lambda_n a^2 \lim_{\varepsilon \rightarrow 0} [X'_n(l - \varepsilon) v(l - \varepsilon, t) + X'_n(\varepsilon) v(\varepsilon, t)] + \\ &- \lambda_n^2 a^2 \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^{l-\varepsilon} v(x, t) X_n(x) dx + \\ &+ \int_{\varepsilon}^{l-\varepsilon} v_t(x, t_0) X_n(x) dx, \quad n = 1, 2, \dots \end{aligned}$$

Bu ifodada (3) bir jinsli chegaraviy shartlarni va $X_n(x)$ funksiyalarning xossalari e'tiborga olib, quyidagi tenglikni olamiz:

$$d'_n(t) = -\lambda_n^2 a^2 \int_0^l v(x, t) X_n(x) dx +$$

$$+ \int_0^l v_t(x, t_0) X_n(x) dx, \quad n = 1, 2, \dots$$

Oxirgi tenglikda (7) belgilashni hisobga olib, quyidagi oddiy differensial tenglamalarga ega bo'lamiz:

$$d'_n(t) + \lambda_n^2 a^2 d_n(t) = d'_n(t_0), \quad n = 1, 2, \dots \quad (10)$$

(10) tenglamaning umumiy yechimi quyidagi ko'rinishga ega:

$$d_n(t) = C_n e^{-a^2 \lambda_n^2 t} + \frac{1}{a^2 \lambda_n^2} (1 - e^{-a^2 \lambda_n^2 t}) d'_n(t_0), \quad n = 1, 2, \dots, \quad (11)$$

bu yerda C_n - ixtiyoriy o'zgarma sonlar.

Endi C_n va $d'_n(t_0)$ koeffitsiyentlarni aniqlaymiz. Buning uchun (11) umumiy yechimni (2) boshlang'ich shartga mos bir jinsli shartlardan, (7) tenglikka asosan kelib chiquvchi, $d_n(0) = 0$ shartga bo'ysundiramiz. Bundan $C_n = 0$, $n = 1, 2, \dots$ ekanligi kelib chiqadi. Natijada (11) umumiy yechim quyidagi ko'rinishni oladi:

$$d_n(t) = \frac{1}{a^2 \lambda_n^2} (1 - e^{-a^2 \lambda_n^2 t}) d'_n(t_0), \quad n = 1, 2, \dots \quad (12)$$

Endi (12) tenglikdan quyidagi tenglikni aniqlaymiz:

$$d'_n(t_0) (1 - e^{-a^2 \lambda_n^2 t_0}) = 0, \quad n = 1, 2, \dots$$

hosil bo'lgan ifodada $1 - e^{-a^2 \lambda_n^2 t_0} \neq 0$, chunki $a > 0$, $t_0 > 0$, $\lambda_n = \frac{n\pi}{l}$, $n \in N$. U holda,

$$d'_n(t_0) = 0, \quad n = 1, 2, \dots$$

tengliklar o'rinli ekanligi kelib chiqadi.

Yuqoridagilarga asosan, (11) umumiy yechim formulasidan $d_n(t) = 0$, .. ekanligi kelib chiqadi. U holda, (7) tenglik quyidagi ko'rinishni oladi:

$$\int_0^l v(x, t) X_n(x) dx = 0, \quad n = 1, 2, \dots$$

Demak, $v(x, t)$ funksiya (6) to'la ortonormal sistemadagi barcha funksiyalar bilan ortogonal ekan. Bu esa Ω sohaning deyarli barcha nuqtalarida $v(x, t) = 0$ ekanligini bildiradi. $u(x, t) \in C_{x,t}^{1,0}(\bar{\Omega}) \cap C_{x,t}^{2,1}(\Omega)$ ekanligidan $v(x, t)$ funksiya $\bar{\Omega}$ sohaning barcha nuqtalarida $v(x, t) \equiv 0$ ekanligi kelib chiqadi.

Buni e'tiborga olsak, (4) tenglikdan $u_1(x, t) = u_2(x, t)$ tenglik kelib chiqadi. Bu esa qo'yilgan masala yechimga ega bo'lsa, u yagona bo'lishini bildiradi. 1-teorema isbotlandi.

Masala yechimining mavjudligi.

2-teorema:

Agar $\varphi(x) \in C^2[0, l] \cap C^3(0, l)$, $\varphi^{(4)}(x) \in L_2(0, l)$, $\varphi(0) = \varphi(l) = 0$, $\varphi''(0) = \varphi''(l) = 0$, shartlar bajarilsa, qo'yilgan masalaning yechimi mavjud.

Isbot. Bu masalani o'zgaruvchilarni ajratish (yoki Furiye) usuli yordamida tadqiq etamiz.

(1) tenglama yechimini

$$u(x, t) = X(x)T(t) \quad (13)$$

ko'rinishda izlaymiz, bu yerda $X(x)$ va $T(t)$ noma'lum funksiyalar. (13) ifodani (1) tenglamaga qo'yib, $X(x)$ va $T(t)$ noma'lum funksiyalarga nisbatan

$$T''(t) + a^2 \lambda T(t) = 0, \quad X''(x) + \lambda X(x) = 0 \quad (14)$$

tenglamalar hosil qilamiz. Bunda $\lambda = \text{const}$. (13) ifodadan va (3) chegaraviy shartlardan

$$X(0) = 0, \quad X(l) = 0 \quad (15)$$

chegaraviy shartlar kelib chiqadi.

(14)-(15) masala xos son va xos funksiyalarni topish haqidagi Shturm-Liuvill masalasidir. (14)-(15) masalaning xos sonlari

$$\lambda_n = \left(\frac{\pi n}{l} \right)^2 \quad (n = 1, 2, \dots)$$

bu xos sonlarga mos trivial bo'lmagan (aynan nolga teng bo'lmagan) normallangan xos funksiyalari

$$X_n(x) = \sqrt{\frac{2}{l}} \sin \frac{\pi n x}{l}, \quad n \in N \quad (16)$$

bo'ladi. Ma'lumki, (16) sistema $L_2(0, l)$ fazoda to'la ortonormal sistemani tashkil etadi.

Endi (1) tenglamaning yechimi

$$u(x, t) = \sum_{n=1}^{\infty} T_n(t) \sqrt{\frac{2}{l}} \sin \frac{\pi n x}{l}, \quad (17)$$

ko'rinishda qidiramiz, (17) ifodani (1) tenglamaga qo'yib va tenglikni ikkala tomonini

$\sqrt{\frac{2}{l}} \sin \frac{\pi n x}{l}$ ga ko'paytirib, $[0, l]$ oralig'ida integrallab

$$T_n'(t) + \left(\frac{a \pi n}{l} \right)^2 T_n(t) = T_n'(t_0) \quad (18)$$

birinchi tartibli yuklangan oddiy differensial tenglama hosil qilamiz. (18) tenglamaning umumiy yechimini o'zgarmasni variatsiyalash usuli yordamida

$$T_n(t) = a_n e^{-a^2 \lambda_n^2 t} + \int_0^t T_n'(t_0) e^{-a^2 \lambda_n^2 (t-\tau)} d\tau \quad (19)$$

ko'rinishda topiladi. (19) tenglikning har ikki tomonidan hosila olib, $t = t_0$ qiymatni hisoblab

$$T_n'(t_0) = \frac{a^2 \lambda_n^2 a_n e^{-a^2 \lambda_n^2 t_0}}{e^{-a^2 \lambda_n^2 t_0} - 1} \quad (20)$$

tenglik hosil qilinadi.

(20) tenglikni (19) tenglikka qo'yib integrallash va soddalashtirish natijasida

$$T_n(t) = a_n e^{-a^2 \lambda_n^2 t} + \left(1 - e^{-a^2 \lambda_n^2 t} \right) \frac{a_n e^{-a^2 \lambda_n^2 t_0}}{1 - e^{-a^2 \lambda_n^2 t_0}} \quad (21)$$

hosil bo'ladi, (21) ni (20) ga qo'yib

$$u(x, t) = \sum_{n=1}^{\infty} \left[a_n e^{-a^2 \lambda_n^2 t} + \left(1 - e^{-a^2 \lambda_n^2 t} \right) \frac{a_n e^{-a^2 \lambda_n^2 t_0}}{1 - e^{-a^2 \lambda_n^2 t_0}} \right] \sqrt{\frac{2}{l}} \sin \frac{\pi n x}{l} \quad (22)$$

qo'yilgan masalaning yechimi hosil qilamiz, bu yerda a_n Furye koeffisienti, (2) boshlang'ich shartdan:

$$u(x, 0) = \sum_{n=1}^{\infty} a_n \sqrt{\frac{2}{l}} \sin \frac{\pi n}{l} x = \varphi(x)$$

kelib chiqadi.

$$a_n = \sqrt{\frac{2}{l}} \int_0^l \varphi(x) \sin \frac{\pi n}{l} x dx. \quad (23)$$

Shunday qilib (1) qo'yilgan masalaning yechimi (22) ko'rinishida aniqlanadi, bu yerda a_n (23) tenglik yordamida aniqlanadi.

(22) qatordan t bo'yicha bir marta, x bo'yicha ikki marta hosila olishdan hosil bo'lgan qatorlar Ω da tekis yaqinlashsa, u holda (22) qatorning yig'indisi qo'yilgan masalaning yechimi bo'ladi.

Lemma. $\forall t \in [0, T]$ va $n = 1, 2, \dots$

bo'lganda quyidagi baholar o'rinli bo'ladi:

$$\begin{aligned} |T_n(t)| &\leq C_1 |a_n|, & |T'_n(t)| &\leq C_2 \lambda_n^2 |a_n|, \\ |T'_n(t_0)| &\leq \lambda_n^2 [|a_n| C_3], \end{aligned} \quad (24)$$

bu yerda C_j , $j = \overline{1, 3}$ - musbat o'zgarmas sonlar.

Isbot. (21) tenglik bilan aniqlanadigan $T_n(t)$ funksiyani baholaymiz. Bunda

$$\begin{aligned} |T_n(t)| &\leq \left| a_n e^{-a^2 \lambda_n^2 t} \right| + \left| \left(1 - e^{-a^2 \lambda_n^2 t} \right) \frac{a_n e^{-a^2 \lambda_n^2 t_0}}{1 - e^{-a^2 \lambda_n^2 t_0}} \right| \leq \\ &\left| a_n e^{-a^2 \lambda_n^2 t} \right| + \left| \frac{a_n e^{-a^2 \lambda_n^2 t_0}}{1 - e^{-a^2 \lambda_n^2 t_0}} \right| = \left| a_n e^{-a^2 \lambda_n^2 t} \right| + \left| \frac{a_n}{e^{a^2 \lambda_n^2 t_0} - 1} \right| \leq \\ &\leq |a_n| + \left| \frac{a_n}{e^{a^2 \left(\frac{\pi}{l} \right)^2 t_0} - 1} \right| = |a_n| \left(1 + \frac{1}{e^{a^2 \left(\frac{\pi}{l} \right)^2 t_0} - 1} \right) = \\ &|a_n| \left(\frac{e^{a^2 \left(\frac{\pi}{l} \right)^2 t_0}}{e^{a^2 \left(\frac{\pi}{l} \right)^2 t_0} - 1} \right) \leq C_1 |a_n|, \end{aligned}$$

Endi (21) $T'_n(t)$ ni hisoblab baholashni ko'rsatamiz:

$$|T'_n(t)| \leq \left| a^2 \lambda_n^2 \left(\frac{e^{-a^2 \lambda_n^2 t_0}}{1 - e^{-a^2 \lambda_n^2 t_0}} - 1 \right) a_n e^{-a^2 \lambda_n^2 t} \right| \leq \lambda_n^2 C_2 |a_n|.$$

$T'_n(t)$ ning ko'rinishidan $T'_n(t_0)$ ning bahosini osonlikcha keltirib chiqarish mumkin. Lemma isbotlandi.

Isbotlangan lemma yordamida dastlab quyidagi qatorning:

$$\frac{\partial u}{\partial t} = - \sum_{n=1}^{\infty} \lambda_n^2 T_n(t) \cdot X_n(x) + \sum_{n=1}^{\infty} T'_n(t_0) \cdot X_n(x) \quad (25)$$

tekis yaqinlashishini ko'rsatamiz. Avval quyidagi

$$\sum_{n=1}^{\infty} \lambda_n^2 T_n(t) X_n(x) \quad (26)$$

qatorni tekshiramiz. Buning uchun (26) qatorga mos mojarant qator tuzamiz.

Lemmadagi (24)

tenglikni e'tiborga olib va o'zgarmaslarni qoldirib yozsak, quyidagi qator hosil bo'ladi:

$$\sum_{n=1}^{\infty} \lambda_n^2 |a_n| \quad (27)$$

ushbu qatorning yaqinlashuvchanligini marta

isbotlash kerak. Buning uchun (23) ni uch bo'laklab integrallab, teoremadagi $\varphi(x)$ funksiyaga qo'yilgan shartlarni inobatga olsak

$$a_n = \sqrt{\frac{2}{l}} \frac{l^3}{(\pi n)^3} \int_0^p \varphi^{(3)}(x) \cos \frac{\pi n}{l} x dx$$

tenglik hosil bo'ladi. Hosil bo'lgan tenglikni

$$a_n = \left(\frac{l}{\pi} \right)^3 \frac{a_n^{(3)}}{n^3} \quad (28)$$

ko'rinishda yozib olamiz, bu yerda

$$a_n^{(3)} = \sqrt{\frac{2}{l}} \int_0^p \varphi_1^{(3)}(x) \cos \lambda_n x dx.$$

(28) ni (27) qatorga qo'yib, o'zgarmaslarni tashlab yuborib

$$\sum_{n=1}^{\infty} \frac{a_n^{(3)}}{n}$$

qatorni hosil qilamiz. Bu qatorning yaqinlashuvchanligi:

$$\left| \frac{a_n^{(3)}}{n} \right| \leq \frac{1}{2} \left[\left(a_n^{(3)} \right)^2 + \frac{1}{n^2} \right]$$

Elementar tengsizlikka asosan kelib chiquvchi

$$\sum_{n=0}^{\infty} \left(a_n^{(3)} \right)^2; \sum_{n=1}^{\infty} \frac{1}{n^2}. \quad (29)$$

qatorlar yaqinlashishidan kelib chiqadi.

(29) ifodadagi birinchi qatorning yaqinlashuvchiligi bo'lakli uzluksiz bo'lgan $\varphi^{(3)}(x)$ funksiya uchun Bessel tengsizligidan kelib chiqadi. Ikkinchisi umumlashgan garmonik $\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}$ qator bo'lib, $\alpha > 1$ bo'lgani uchun yaqinlashuvchi bo'ladi [5]. Demak, (26) qator yaqinlashadi. Endi

$$\sum_{n=1}^{\infty} T'_n(t_0) \cdot X_n(x) \quad (30)$$

qatorni tekis yaqinlashuvchilikka tekshiramiz. (30) qatorning yaqinlashishini ham yuqoridagidek tekshirish mumkin. Buning uchun unga mos

$$\sum_{n=1}^{\infty} |T'_n(t_0)|$$

mojarant qator tuziladi. Lemmadagi (24) tenglikni e'tiborga olib va o'zgarmalarni qoldirib, quyidagi qator hosil bo'ladi:

$$\sum_{n=1}^{\infty} \lambda_n^2 |a_n|$$

Bu qatorning yaqinlashuvchanligi (27) tenglikni 3 marta bo'laklab, hosil bo'lgan tenglikni soddalashtirib, o'zgarmaslarni tashlab yuborib:

$$\sum_{n=1}^{\infty} \frac{a_n^{(3)}}{n}$$

qatorni hosil qilamiz. Bu qatorning ham yaqinlashishini (29) qatorlar yordamida isbotlash mumkin. (26) va (30) qatorlarning tekis yaqinlashuvchanligidan (25) qatorni ham tekis yaqinlashuvchanligi kelib chiqadi. Demak (25) qator ham tekis yaqinlashadi. Teorema isbotlandi.

Foydalanilgan adabiyotlar

- [1]. Nakhushiev, A.M. Loaded Equations in Applied Problems. Moscow: Nauka, 1995.
- [2]. Borisov, A.V., Kreferov, A.G. Boundary value problems for loaded parabolic equations // Differential Equations 1999, Vol. 35, No. 2, pp. 210–219.
- [3]. Agarwal, R.P., Baltaeva, Z.A., Khasanov, A.K., Existence and uniqueness of the solution to initial and inverse problems for integro-differential heat equations with fractional load // Electronic Journal of Differential Equations, 2024, No. 64, pp. 1–19.
- [4]. Ceretani, A., Tarzia, D.A., Villa, R. Explicit solutions for non-classical heat conduction problems with source depending on the boundary flux // ArXiv preprint, arxiv:1406.4295, 2014.
- [5]. Azlarov T., Mansurov X. Matematik analiz. 2-qism. Toshkent, O'qituvchi 1989.

